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BIALGEBRAS AND GALOIS EXTENSIONS

Dedicated to Professor Takasi Nagahara on his 60th birthday

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We shall work over a commutative ring k. A *bialgebra* $B=(B, \mu, \eta, \Delta, \varepsilon)$ is simultaneously a k-algebra with multiplication μ and unit η, and a k-coalgebra with comultiplication Δ and counit ε which are algebra morphisms. A bialgebra B is called a *left Hopf algebra* if it has a morphism $\lambda: B \to B$ such that

$$\sum_{(x)}\lambda(x_{(1)})x_{(2)} = \mu(\lambda \otimes Id)\Delta(x) = \varepsilon(x) \quad (x \in B),$$

where $\otimes = \otimes_k$, $\Delta(x) = \sum_{(x)} x_{(1)} \otimes x_{(2)}$ and Id is the identity morphism. λ is called a *left antipode* ([5], [10]). A right Hopf algebra is similarly defined. A left and right Hopf algebra is a Hopf algebra in the usual sense ([12]).

Let S be a k-algebra. S is called a *right B-comodule algebra* if there exists a k-algebra morphism $\alpha_S: S \to S \otimes B$ such that $(\alpha_S \otimes Id)\alpha_S = (Id \otimes \Delta)\alpha_S$ and $(Id \otimes \varepsilon)\alpha_S = Id$. The *$k$-subalgebra of invariants* is defined by $R = S_0 = \{s \in S \mid \alpha_S(s) = s \otimes 1\}$. According to Y. Doi and M. Takeuchi [4], we call that S/R is a *right B-extension.* A left B-extension is defined by using a left comodule algebra instead of the right. A right B-extension S/R is called a *right B-Galois extension* if the morphism $\gamma_S: S \otimes_R S \to S \otimes B$ defined by $\gamma_S(s \otimes t) = \sum_{(t)} st_{(0)} \otimes t_{(1)}$ is bijective, where $\alpha_S(t) = \sum_{(t)} t_{(0)} \otimes t_{(1)}$. Since B is a right B-comodule algebra with the structure morphism Δ and the invariant k-subalgebra is $B_0 = k$, B/k is a right B-extension. Then by [3, Cor. 6], B is a right B-Galois extension if and only if B has the antipode (that is, B is a Hopf algebra).

In this paper we consider a right B-Galois extension for a bialgebra B which is not a Hopf algebra. In section 1, we discuss a relation of right and left antipode of a bialgebra B to a right B-Galois extension. In section 2, we assume that there exists a right B-Galois extension S/R. Under the condition, we give some sufficient conditions that a bialgebra B becomes a Hopf algebra. And in section 3, if B is a semigroup bialgebra which is generated by a finite cyclic semigroup (not a group), then there does not exist

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a right B-Galois extension and a right $B^* = \mathrm{Hom}_k(B, k)$-Galois extension of k.

Throughout the following, we will fix the above notations and terminologies. All algebras, morphisms, $\otimes$, Hom, etc. are taken over k unless otherwise stated and we freely use a sigma notation in the sense of M. Sweedler [12].

1. The antipode and the morphism γ_B. We discuss a relation of a right (resp. a left) antipode to a left (resp. a right) invertibility of γ_B.

Theorem 1.1. *Let B be a bialgebra and let γ_B be the morphism from $B \otimes B$ to $B \otimes B$ defined by $\gamma_B(x \otimes y) = \sum_{(y)} xy_{(1)} \otimes y_{(2)}$. Then*

(1) *B has a right antipode if and only if γ_B has a left inverse.*

(2) *If B has a left antipode, then γ_B has a right inverse.*

(3) *If γ_B is an isomorphism, then B has a left antipode and thus B has the antipode.*

Proof. (1) Let λ be a right antipode. We define a morphism β: $B \otimes B \to B \otimes B$ by $\beta(x \otimes y) = \sum_{(y)} x\lambda(y_{(1)}) \otimes y_{(2)}$ $(x, y \in B)$. Then

$$\beta\gamma_B(x \otimes y) = \sum_{(y)} xy_{(1)}\lambda(y_{(2)}) \otimes y_{(3)} = \sum_{(y)} x\varepsilon(y_{(1)}) \otimes y_{(2)} = x \otimes y,$$

and so $\beta\gamma_B = Id$.

Conversely, let β: $B \otimes B \to B \otimes B$ be a left inverse of γ_B. Since β is a left B-module morphism, we can set $\beta(1 \otimes x) = \sum_i x_{1i} \otimes x_{2i}$. By $1 \otimes x = \beta\gamma_B(1 \otimes x) = \sum_{(x),i} x_{(1)}x_{(2)1i} \otimes x_{(2)2i}$, we have

$$\varepsilon(x) = \sum_{(x),i} x_{(1)}x_{(2)1i}\varepsilon(x_{(2)2i}).$$

Define a morphism λ: $B \to B$ by $\lambda(x) = \sum_i x_{1i}\varepsilon(x_{2i})$. Then it is easy to see that

$$\mu(Id \otimes \lambda)\Delta(x) = \sum_{(x),i} x_{(1)}x_{(2)1i}\varepsilon(x_{(2)2i}) = \varepsilon(x).$$

Thus λ is a right antipode.

(2) If λ is a left antipode of B, then the morphism β: $B \otimes B \to B \otimes B$ defined by $\beta(x \otimes y) = \sum_{(y)} x\lambda(y_{(1)}) \otimes y_{(2)}$ satisfies the relation $\gamma_B\beta = Id$.

(3) Assume that β is the inverse morphism of γ_B. For any $f, g \in \mathrm{Hom}(B, B)$, we define morphisms

$$\Phi, \Psi: \mathrm{Hom}(B, B) \to \mathrm{Hom}(B, B)$$

by

$$\Phi(f)(x) = \sum x_{(1)} f(x_{(2)}) \quad \text{and} \quad \Psi(g)(x) = \sum_i x_{1i} g(x_{2i}),$$

where $\beta(1 \otimes x) = \sum_i x_{1i} \otimes x_{2i}$. By $\gamma_B \beta = Id$, we have

$$(1.1) \qquad 1 \otimes x = \gamma_B \beta(1 \otimes x) = \sum_{(x_{2i}),i} x_{1i} x_{2i(1)} \otimes x_{2i(2)}.$$

Then by (1.1),

$$((\Psi\Phi)(f))(x) = \sum_i x_{1i} \Phi(f)(x_{2i}) = \sum_{(x_{1i}),i} x_{1i} x_{2i(1)} f(x_{2i(2)}) = f(x)$$

and so $\gamma_B \beta = Id$ implies $\Psi\Phi = Id$. Conversely if $\beta\gamma_B = Id$, then

$$1 \otimes x = \beta\gamma_B(1 \otimes x) = \sum_{(x),i} x_{(1)} x_{(2)1i} \otimes x_{(2)2i},$$

which shows that

$$(1.2) \qquad g(x) = \sum_{(x),i} x_{(1)} x_{(2)1i} g(x_{(2)2i}).$$

Therefore

$$((\Phi\Psi)(g))(x) = \sum_{(x)} x_{(1)} \Psi(g)(x_{(2)}) = \sum_{(x),i} x_{(1)} x_{(2)1i} g(x_{(2)2i}) = g(x).$$

Now we define $\lambda : B \to B$ by $\lambda(x) = \sum_i x_{1i} \varepsilon(x_{2i})$, and consider the morphism $\mu(\lambda \otimes Id)\Delta$ in Hom(B, B). An easy calculation shows

$$\Phi(\mu(\lambda \otimes Id)\Delta)(x) = \sum_{(x)} x_{(1)} \mu(\lambda \otimes Id)\Delta(x_{(2)}) = \sum_{(x),i} x_{(1)} x_{(2)1i} \varepsilon(x_{(2)2i}) x_{(3)}.$$

In(1.2), if we take $g = \varepsilon$, then

$$\varepsilon(x) = \sum_{(x),i} x_{(1)} x_{(2)1i} \varepsilon(x_{(2)2i}).$$

Thus

$$\begin{aligned} \Phi(\mu(\lambda \otimes Id)\Delta)(x) &= \sum_{(x),i} x_{(1)} x_{(2)1i} \varepsilon(x_{(2)2i}) x_{(3)} = \sum_{(x)} \varepsilon(x_{(1)}) x_{(2)} \\ &= x = \sum_{(x)} x_{(1)} \varepsilon(x_{(2)}) = \Phi(\varepsilon)(x). \end{aligned}$$

Since $\Psi\Phi = Id$, we have $\mu(\lambda \otimes Id)\Delta = \varepsilon$. This shows that λ is a left antipode and by (1), λ is itself a right antipode.

By Th. 1.1, B is a right B-Galois extension of k if and only if B has the antipode. This was proved by Y. Doi and M. Takeuchi in [3, Cor. 6].

By the similar proof to Th. 1.1, we have the following:

Theorem 1.2. *Let B be a bialgebra and let ${}_B\gamma$ be the morphism from $B\otimes B$ to $B\otimes B$ defined by ${}_B\gamma(x\otimes y)=\Delta(x)(1\otimes y)=\sum_{(x)}x_{(1)}\otimes x_{(2)}y$. Then*

(1) *B has a left antipode if and only if ${}_B\gamma$ has a left inverse.*
(2) *If B has a right antipode, then ${}_B\gamma$ has a right inverse.*
(3) *B has the antipode if and only if ${}_B\gamma$ has a two-sided inverse.*

For finitely generated k-modules M and N, if $f\colon M\to N$ is an epimorphism, then by [11, Th. 1], f is an isomorphism. Therefore by Th. 1.1(2) and Th. 1.2(2), we have the following result which was proved by J. A. Green, W. D. Nichols and E. J. Taft in [5, Prop. 5].

Corollary 1.3. *Let B be a left (or right) Hopf algebra. If B is a finitely generated k-module, then B is a Hopf algebra.*

2. Quadratic bialgebras and Galois extensions. In [7], H. K. Kreimer determined a Hopf algebra which is a free k-module of rank 2 as follows: Let B be a k-bialgebra which is a free k-module of rank 2. Since $\varepsilon: B\to k$ is a k-module epimorphism, we have $B=k\oplus Ker(\varepsilon)$. In general it is known that M is a free k-module of rank 1 if and only if $k\oplus M$ is a free k-module of rank 2 (cf. [7, Lemma 1]). Hence there exists a free basis $\{1,\theta\}$ of B such that $Ker(\varepsilon)=k\theta$. Since $\varepsilon(\theta^2)=\varepsilon(\theta)^2=0$, $\theta^2=q\theta$ for some $q\in k$. And by $\theta=(\varepsilon\otimes 1)\Delta(\theta)=(1\otimes\varepsilon)\Delta(\theta)$, $\Delta(\theta)=\theta\otimes 1+1\otimes\theta+p(\theta\otimes\theta)$ for some $p\in k$. Moreover by $\Delta(\theta^2)=\Delta(\theta)^2$ and $\theta^2=q\theta$, we have

$$p^2q^2+3pq+2=(pq+1)(pq+2)=0. \tag{2.1}$$

If B has the antipode λ, then by definition of λ and $\theta^2=q\theta$, there exists $h\in k$ such that $\lambda(\theta)=h\theta$ and $(-h)(pq+1)=1$. Therefore

(2.2) B has the antipode if and only if $pq+2=0$.

Now we will discuss the following question. For a bialgebra B (not a Hopf algebra) which is a free k-module of rank 2, does there exist a right B-Galois extension of k?

Let S be a commutative B-comodule algebra with comodule structure morphism $\alpha\colon S\to S\otimes B$. Let $R=S_0$ be the invariant k-subalgebra of S. We assume that S is a free R-module with basis $\{1,x\}$ and set $x^2=mx+n$

$(m, n \in R)$. Noting that R is invariant under α, α can be considered as an R-module morphism. Then by $(Id \otimes \varepsilon)\alpha = Id$, we can set $\alpha(x) = x \otimes 1 + c(1 \otimes \theta) + d(x \otimes \theta)$ for some $c, d \in R$. Since $(\alpha \otimes Id)\alpha = (Id \otimes \Delta)\alpha$ and α is an R-algebra morphism, we have

$$cd = cp \tag{2.3}$$
$$d^2 = dp \tag{2.4}$$
$$cm = c^2q + d^2nq + 2dn \tag{2.5}$$
$$dm = d^2mq + 2(c + cqd + dm) \tag{2.6}$$

Then, for the ordered basis $\{1 \otimes 1, x \otimes 1, 1 \otimes x, x \otimes x\}$ of $S \otimes_R S$ and $\{1 \otimes 1, x \otimes 1, 1 \otimes \theta, x \otimes \theta\}$ of $S \otimes B$, the matrix representation of the morphism $\gamma_S : S \otimes_R S \to S \otimes B$ is given by

$$A = \begin{pmatrix} 1 & 0 & 0 & n \\ 0 & 1 & 1 & m \\ 0 & 0 & c & dn \\ 0 & 0 & d & c+dm \end{pmatrix}$$

and the determinant of A is $c(c+dm) - d^2n$. Therefore

γ_S is an isomorphism if and only if $c(c+dm) - d^2n$ is invertible in R.

Now we assume that S/k is a right B-Galois extension. Noting that (2.3), (2.4), (2.5) and the fact that $c(c+dm) - d^2n$ is invertible, we can easily see that

$$\begin{aligned} c(c+dm) - d^2n &= c^2 + cpm - d^2n = c^2 + p(c^2q + d^2nq + 2dn) - d^2n \\ &= c^2(1+pq) + d^2npq + d^2n \\ &= (c^2 + d^2n)(1+pq). \end{aligned}$$

Therefore $1 + pq$ is invertible and by (2.1), $pq + 2 = 0$. This means that B has the antipode. These proves the following

Theorem 2.1. *Let B be a bialgebra which is a free k-module of rank 2. Let S be a commutative right B-comodule algebra with invariant k-subalgebra R. If S/R is a right B-Galois extension which is a free R-module of rank 2, then B has the antipode.*

Using the localization, we have

Corollary 2.2. *Let B and S be as in Th. 2.1. If a right B-Galois extension S/R is a finitely generated projective k-module of rank 2, then B has the antipode.*

Next we consider a non-commutative quadratic extension. Let k be a commutative ring generated by 1 and let S be a k-algebra with subalgebra R. Assume that S is a free quadratic extension of R, that is, $S = R \oplus Rx$ and $x^2 = mx+n$ $(m, n \in R)$, where $\{1, x\}$ is a free basis. We set

$$xr = \sigma(r)x+D(r) \quad (r \in R).$$

Then it is easy to see that σ is a k-algebra morphism with $\sigma(1) = 1$ and $D(st) = D(s)t+\sigma(s)D(t)$ $(s, t \in R)$. If S is a right B-comodule algebra with structure morphism α: $S \to S \otimes B$ and $S_0 = R$, then α is a left R-linear morphism which is given by

$$\alpha(x) = x \otimes 1+c \otimes \theta+dx \otimes \theta \quad (c, d \in R),$$

and the relation (2.3) and (2.4) are also satisfied in the same way as to the commutative case. Since α is a k-algebra morphism, we have the following equations

(2.7) $mc = (1+dq)D(c)+\{d+\sigma(d)+d\sigma(d)q\}n+c^2q,$

(2.8) $md = (1+dq)\{c+\sigma(c)+D(d)\}+\{d+\sigma(d)+d\sigma(d)q\}m,$

(2.9) $dD(r) = \sigma(r)c-cr \quad (r \in R),$

(2.10) $d\sigma(r) = \sigma(r)d \quad (r \in R).$

Assume that S is a right B-Galois extension of R. Then γ_S: $S \otimes_R S \to S \otimes B$ is an epimorphism if and only if there exists elements r_0, r_1, r_2, r_3 in R such that

(2.11) $r_0+r_3n = 0,$

(2.12) $r_1+r_2+r_3m = 0,$

(2.13) $r_2c+r_3\{D(c)+\sigma(d)n\} = 1,$

(2.14) $r_2d+r_3\{\sigma(c)+\sigma(d)m+D(d)\} = 0.$

Under these notations, we have the following

Theorem 2.3. *Assume that there exists a right B-Galois extension S/R. If one of the following assumptions satisfies, then B is a Hopf algebra.*

(1) $c = 0$ *and* σ *is an epimorphism.*

(2) c *is not a right zero-divisor in* R.

(3) R *has no right zero-divisors.*

(4) $d = p$, *or* $d = 0$.

Proof. In the following proof, we note that p and q are in the center of R and they are invariant under σ and D.

(1) If $c=0$, then by (2.13), $r_3\sigma(d)n=1$. Since σ is an epimorphism, d is in the center of R by (2.10) and so $\sigma(d)$ is invertible in R. Thus by (2.4), $\sigma(d)=p$. By (2.7) and the fact that p and $\sigma(n)$ have left inverse, we get $pq+2=0$, which shows that B is a Hopf algebra by (2.2).

(2) If c is not a right zero-divisor, then by (2.3) $d=p$. Using this and (2.8), we have

$$(2.15) \qquad (pq+1)\{c+\sigma(c)+pm\}=0.$$

Since γ_S is a monomorphism, (2.11), (2.12), (2.14) and

$$(2.16) \qquad r_2c+r_3\{D(c)+\sigma(d)n\}=0$$

imply that $r_0=r_1=r_2=r_3=0$, and by (2.14) and (2.16), we get that

$$(2.17) \qquad pD(c)+p^2n-\sigma(c)c-pmc \text{ is not a right zero-divisor.}$$

Then noting that (2.7), (2.9) and (2.17), we can prove that

$$(2.18) \qquad (pq+1)\{p^2n+\sigma(c)c\} \text{ is not a right zero-divisor.}$$

Thus by (2.15), $pm=-c-\sigma(c)$. Multiplying (2.7) by p and using (2.9), we get $(pq+2)\{p^2n+\sigma(c)c\}=0$ and by (2.18), $pq+2=0$.

(3) By (2), we may assume that $c=0$. If $d=0$, then α is the identity morphism and S is not a right B-Galois extension of R. If $d\neq 0$, then $d=p$ and by (2.9) $D=0$. Since γ_S is an epimorphism, $r_3pn=1$ by (2.13). Moreover by (2.7), $(pq+2)pn=0$ and so $pq+2=0$.

(4) If $d=p$, then the result is clear by the proof of (2). If $d=0$, then by (2.7) and (2.13), we get $r_2c+r_3D(c)=(r_2+r_3m-r_3cq)c=1$, which shows that c has a left inverse element. Thus by (2), B is a Hopf algebra.

In [6], S. Ikehata got some results for an ideal generated by a single polynomial in a skew polynomial ring and in [9], T. Nagahara studied many type of skew polynomials of degree 2 in detail with respect to non-commutative Galois extensions. And there are many papers for quadratic extensions and skew polynomial rings. But in any cases it is not known that there exists a quadratic free right B-Galois extension of R. Because our type of right B-Galois extensions is beyond the scope of the recent work of quadratic extensions and skew polynomial rings.

Question. Does there exist a free quadratic right B-Galois extension of R?

3. The case of finite cyclic semigroup. Let G be a finite cyclic semigroup (not a group) with identity 1, that is,

$$G = \{1, \sigma, \sigma^2, \dots, \sigma^{n-1} \mid \sigma^n = \sigma^m \text{ for some } 1 \leq m \leq n-1\}.$$

Let $B = kG$ be a semigroup bialgebra with usual coalgebra structure $\Delta(\sigma) = \sigma \otimes \sigma$ and $\varepsilon(\sigma) = 1$, S a commutative right B-Galois extension of k, and $\alpha: S \to S \otimes B$ a comodule algebra structure morphism. According to the same method in [1, pp. 35—39], we can determine the structure of S as follows : First, we can set that

$$\alpha(s) = \sum_{i=0}^{n-1} s_i \otimes \sigma^i \ (s, s_i \in S).$$

Using the comodule structure of S, it is easy to see that

$$s = \sum_{i=0}^{n-1} s_i \quad \text{and} \quad \alpha(s_i) = s_i \otimes \sigma^i.$$

We set $S_i = \{s \in S \mid \alpha(s) = s \otimes \sigma^i\}$. Then S_i is a k-module and $S = S_0 \oplus \cdots \oplus S_{n-1}$ as a k-module and $S_0 = k$. Since $\gamma_S: S \otimes S \to S \otimes B$ is an epimorphism, there exist $s_i, t_i \in S_i$ such that

$$\beta_S\left(\sum_{i,j=0}^{n-1} s_i \otimes t_j\right) = \sum_{i,j=0}^{n-1} s_i t_j \otimes \sigma^j = 1 \otimes \sigma.$$

Therefore $(\sum_{i=0}^{n-1} s_i) t_1 = 1$. By $\alpha(s_i t_j) = \alpha(s_i)\alpha(t_1) = s_i t_1 \otimes \sigma^{i+1}$, $s_i t_1$ are contained in S_{i+1} and so $1 \in S_1 \oplus S_2 \oplus \cdots \oplus S_{n-1}$, because $S_n = S_m$. This contradicts to $S_0 \cap (S_1 \oplus S_2 \oplus \cdots \oplus S_{n-1}) = 0$. Thus there does not exist a right kG-Galois extension of k for a cyclic semigroup G.

Next let B be the dual bialgebra of kG, that is, $B = \mathrm{Hom}(kG, k)$, S/k a right B-extension and $\alpha: S \to S \otimes B$ a comodule algebra structure morphism. Then by the coalgebra structure of kG, G acts on S as a k-algebra morphism and

$$\alpha(s) = \sum_{i=0}^{n-1} \sigma^i(s) \otimes v_i,$$

where $\{v_0, v_1, \dots, v_{n-1}\}$ are the ordered dual basis of B with respect to the basis $\{1, \sigma, \dots, \sigma^{n-1}\}$ of kG (cf. [8, Example 1]). Assume that S/k is a right B-Galois extension. Then $\gamma_S: S \otimes S \to S \otimes B$ is an epimorphism and so there exist $s_i, t_i \in S$ such that

$$\gamma_S\left(\sum_{i=0}^{r} s_i \otimes t_i\right) = \sum_{i=0}^{r} \sum_{j=0}^{n-1} s_i \sigma^j(t_i) \otimes v_j = 1 \otimes v_0.$$

Therefore

$$\sum_{i=0}^{r} s_i t_i = 1 \quad \text{and} \quad \sum_{i=0}^{r} s_i \sigma^j(t_i) = 0$$

for any $j = 1, 2, \ldots, n-1$. Since $\sigma^n = \sigma^m$ and σ is a k-algebra morphism, we have the following contradiction :

$$0 = \sigma^m\left(\sum_{i=0}^{r} s_i \sigma^{n-m}(t_i)\right) = \sum_{i=0}^{r} \sigma^m(s_i)\,\sigma^n(t_i)$$

$$= \sum_{i=0}^{r} \sigma^m(s_i)\,\sigma^m(t_i) = \sigma\left(\sum_{i=0}^{r} s_i t_i\right) = \sigma^m(1) = 1.$$

Thus S/k is not a right B-Galois extension. These prove the following

Theorem 3.1. *Let G be a finite cyclic semigroup (not a group) and let B be the semigroup bialgebra kG or $(kG)^* = \mathrm{Hom}(kG, k)$. Then there does not exist a right B-Galois extension of k.*

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