Math. J. Okayama Univ. 32 (1990), 207-221

ON REAL HYPERSURFACES OF A COMPLEX SPACE FORM

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Introduction. A complex n-dimensional Kaehler manifold of constant holomorphic sectional curvature c is called a complex space form, which is denoted by $M_n(c)$. Let F be its complex structure. The complete and simply connected complex space form consists of a complex projective space CP^n, a complex Euclidean space C^n or a complex hyperbolic space CH^n, according as $c>0$, $c=0$ or $c<0$.

In the study of real hypersurfaces of a complex projective space CP^n, Takagi [11] classified all homogeneous real hypersurfaces of CP^n. He showed also that real hypersurfaces of CP^n with 2 or 3 distinct constant principal curvatures are homogeneous.

On the other hand, Cecil and Ryan [2] studied pseudo-Einstein real hypersurfaces of CP^n on which $\xi=-FC$ is principal, where C is the unit normal vector field on M. They showed that if ξ is principal, then M lies on a tube over a Kaehler submanifold. By making use of this notion and the results of Takagi's classification, Kimura [3] proved the following.

Theorem A. *Let M be a connected real hypersurface of CP^n. Then M has constant principal curvatures and ξ is principal if and only if M is locally congruent to one of the following*

(A_1) *a tube over a hyperplane CP^{n-1}.*

(A_2) *a tube over a totally geodesic $CP^k (1 \leqq k \leqq n-2)$.*

(B) *a tube over a complex quadric Q_{n-1}.*

(C) *a tube over $CP^1 \times CP^{(n-1)/2}$ and $n(\geqq 5)$ is odd.*

(D) *a tube over a complex Grassmann $G_{2,5}(C)$ and $n=9$.*

(E) *a tube over a Hermitian symmetric space $SO(10)/U(5)$, and $n=15$.*

According to Takagi's classification [11], the principal curvatures and their multiplicities of the above homogeneous real hypersurfaces are given.

On the other hand, real hypersurfaces of a complex hyperbolic space CH^n have also been investigated by Berndt [1], Montiel [8], Montiel and

*This research was partially supported by KOSEF.

Romero [9]. In particular, by using the notion of the tube in Cecil and Ryan [2], Montiel [8] also classified the real hypersurface of complex hyperbolic space with at most two distinct principal curvatures. Recently, Berndt [1] classified all real hypersurfaces with constant principal curvature of CH^n under the condition such that ξ is principal. Namely he proved the following.

Theorem B. *Let M be a connected real hypersurface of $CH^n(n \geqq 2)$. Then M has constant principal curvatures and ξ is principal if and only if M is locally congruent to one of the following*

(A_1) *a horosphere in CH^n.*

(A_2) *a tube over CH^k for a $k = 0, 1, \ldots, n-1$.*

(B) *a tube over RH^n.*

For the principal curvatures and their multiplicities of the above hypersurfaces are also given in [1].

The purpose of this paper is to characterize some real hypersurfaces of $M_n(c)$, $c \neq 0$, by using above classification theorems. The authors would like to express their thanks to the referee for his valuable comments.

1. Preliminaries. Let M be a real hypersurface of a complex n dimensional complex space form $M_n(c)$, and let C be a unit normal vector field on a neighborhood of a point x in M. Let us denote by F the almost complex structure of $M_n(c)$. For any local vector field X on a neighborhood of x in M, the transformations of X and C under F can be given by

$$FX = \phi X + \eta(X)C, \; FC = -\xi,$$

where ϕ defines a skew-symmetric transformation on the tangent bundle TM of M, while η and ξ denote a 1-form and a vector field on a neighborhood of X in M respectively. Then it is seen that $g(\xi, X) = \eta(X)$, where g denotes the induced Riemannian metric on M. The set of tensors (ϕ, ξ, η, g) is called an almost contact structure on M. They satisfy the following

$$\phi^2 = -I + \eta \otimes \xi, \; \phi\xi = 0, \; \eta(\phi X) = 0, \; \eta(\xi) = 1, \tag{1.1}$$

where I denotes the identity transformation. Furthermore, the covariant derivatives of the structure tensors are given by

$$(\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi, \; \nabla_X \xi = \phi AX, \tag{1.2}$$

where ∇ is the Riemannian connection of g and A denotes the shape operator with respect to the unit normal C on M.

Since the ambient space is of constant holomorphic sectional curvature c, the equations of Gauss and Codazzi are respectively given as follows

$$(1.3)\quad \begin{aligned} R(X, Y)Z = c\{ & g(Y, Z)X - g(X, Z)Y + g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y \\ & -2g(\phi X, Y)\phi Z\}/4 + g(AY, Z)AX - g(AX, Z)AY, \end{aligned}$$

$$(1.4)\quad (\nabla_X A)Y - (\nabla_Y A)X = c\{\eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi\}/4,$$

where R denotes the Riemannian curvature tensor of M and $\nabla_X A$ denotes the covariant derivative of the shape operator A with respect to X.

The Ricci tensor S' of M is the tensor of type $(0, 2)$ given by $S'(X, Y) = \mathrm{tr}\{Z \to R(Z, X)Y\}$. Also it may be regarded as the tensor of type $(1,1)$ and denoted by $S : TM \to TM$; it satisfies $S'(X, Y) = g(SX, Y)$. From (1.3) we see that the Ricci tensor S of M is given by

$$(1.5)\quad S = c\{(2n+1)I - 3\eta \otimes \xi\}/4 + hA - A^2,$$

where we have put $h = \mathrm{tr}A$. A real hypersurface M of $M_n(c)$ is said to be pseudo-Einstein if the Ricci tensor S satisfies

$$(1.6)\quad SX = aX + b\eta(X)\xi$$

for any vector field X tangent to M and some functions a and b on M.

2. Certain Lemmas. Let M be a real hypersurface of a complex space form $M_n(c)$. The shape operator A of M can be considered as a symmetric $(2n-1, 2n-1)$-matrix. Now we assume that the structure vector ξ is an eigenvector of A, that is, $A\xi = \alpha\xi$. Then the second formula of (1.2) gives

$$(\nabla_X A)\xi = (X\alpha)\xi + \alpha\phi AX - A\phi AX,$$

from which it follows that

$$(2.1)\quad g((\nabla_X A)Y, \xi) = (X\alpha)\eta(Y) + \alpha g(Y, \phi AX) - g(Y, A\phi AX).$$

By using equation of Codazzi to (2.1) we have

$$(2.2)\quad \begin{aligned} cg(X, \phi Y)/2 = & (X\alpha)\eta(Y) - (Y\alpha)\eta(X) + \alpha g((\phi A + A\phi)X, Y) \\ & - 2g(A\phi AX, Y). \end{aligned}$$

Putting $X = \xi$ or $Y = \xi$ in (2.2), then we see that $X\alpha = (\xi\alpha)\eta(X)$, or $Y\alpha = (\xi\alpha)\eta(Y)$ and hence (2.2) reduces to

$$(2.3)\quad cg(X, \phi Y)/2 = \alpha g((\phi A + A\phi)X, Y) - 2g(A\phi AX, Y).$$

First of all we prove the following.

Lemma 2.1. *Let M be a real hypersurface of a complex space form $M_n(c)$. If $\phi A+A\phi=0$, then $c=0$.*

Proof. By the assumption we have that ξ is an eigenvector of A. From this assumption, and the almost contact structure (ϕ, ξ, η, g) and (2.3) it follows that

$$A^2=-cI/4+(\alpha^2+c/4)\eta\otimes\xi. \tag{2.4}$$

We notice here that the holomorphic sectional curvature c is non-positive. Thus (2.1) and (2.3) imply

$$g((\nabla_X A)Y, \xi)=-cg(Y, \phi X)/4+\alpha g(Y, \phi AX)+(X\alpha)\eta(Y). \tag{2.5}$$

Differentiating (2.4) covariantly along M, we find

$$\begin{aligned}(\nabla_X A)AY+A((\nabla_X A)Y)=&\,2\alpha(\xi\alpha)\eta(X)\eta(Y)\xi+(\alpha^2+c/4)\\ &\times\{g(\nabla_X\xi, Y)\xi+\eta(Y)\nabla_X\xi\},\end{aligned} \tag{2.6}$$

from which, taking the skew-symmetric part and using the equation of Codazzi, we have

$$(\nabla_X A)AY-(\nabla_Y A)AX=c\alpha g(\phi X, Y)\xi/2+\alpha^2\{\eta(Y)\phi AX-\eta(X)\phi AY\},$$

where we have used the fact $\phi A+A\phi=0$. Equivalently it follows that

$$\begin{aligned}g(AY,(\nabla_Z A)X)-g(AZ,(\nabla_Y A)X)=&\,c\alpha\eta(X)g(\phi Z, Y)/2\\ &+\alpha^2\{\eta(Y)g(\phi AZ, X)-\eta(Z)g(\phi AY, X)\},\end{aligned}$$

from which, also using the equation of Codazzi, we have

$$\begin{aligned}g(AY,(\nabla_X A)Z)-g(AZ,(\nabla_X A)Y)=&\,c\alpha/2\{\eta(X)g(Y, \phi Z)\\ &+\eta(Y)g(X, \phi Z)+\eta(Z)g(Y, \phi X)\}\\ &+(\alpha^2-c/4)\{\eta(Y)g(\phi AZ, X)-\eta(Z)g(\phi AY, X)\}.\end{aligned} \tag{2.7}$$

Summing up (2.6) and (2.7), we have

$$\begin{aligned}(\nabla_X A)AY=&\,\alpha(\xi\alpha)\eta(X)\eta(Y)\xi+\alpha^2\eta(Y)\phi AX+cg(\phi AY, X)\xi/4\\ &+c\alpha\{-\eta(X)\phi Y-\eta(Y)\phi X+g(Y, \phi X)\xi\}/4.\end{aligned}$$

From which, substituting AY into Y and using (2.4) and (2.5), we have that

$$\begin{aligned}c(\nabla_X A)Y=&\,c(\xi\alpha)\eta(X)\eta(Y)\xi+c^2\{g(\phi Y, X)\xi-\eta(Y)\phi X\}/4\\ &+c\alpha\{\eta(X)\nabla_Y\xi+g(\nabla_Y\xi, X)\xi+\eta(Y)\nabla_X\xi\}.\end{aligned} \tag{2.8}$$

Now we take an orthonormal frame $\{E_i\}$ of $T_x(M)$ such that $\nabla_{E_i}E_j=0$ $(i, j, \ldots, =1, 2, \ldots, 2n-1)$. Differentiating (2.8) with respect to E_i and using the fact that $E_i\alpha = (\xi\alpha)\eta(E_i)$, it follows from the almost contact structure that we have

$$(2.9)\quad \begin{aligned} \sum_{i,j} cg(\phi E_i, E_j)(\nabla_{E_i}\nabla_{E_j}A)Y &= \frac{c^2}{4}\sum_i \{g(\phi Y, \phi E_i)\nabla_{E_i}\xi - g(\nabla_{E_i}\xi, Y)\phi^2 E_i\} \\ &+ c\alpha\sum_i \{g(\nabla_{E_i}\xi, \phi E_i)\nabla_Y\xi + g(\nabla_{E_i}\nabla_Y\xi - \nabla_{\nabla_{E_i}Y}\xi, \phi E_i)\xi \\ &+ g(\nabla_{E_i}\xi, Y)\nabla_{\phi E_i}\xi + g(\nabla_Y\xi, \phi E_i)\nabla_{E_i}\xi + \eta(Y)\nabla_{E_i}\nabla_{\phi E_i}\xi\}. \end{aligned}$$

On the other hand, we have

$$\sum_i g(\nabla_{E_i}\xi, \phi E_i) = \sum_i \{g(AE_i, E_i) - g(AE_i, \eta(E_i)\xi)\} = 0,$$

where in the last step we have used the fact that the mean curvature of A coincides with α because of the assumption $A\phi + \phi A = 0$. From the almost contact structure and the fact ξ is principal the following formula also vanishes.

$$\sum_i \{g(\nabla_{E_i}\xi, Y)\nabla_{\phi E_i}\xi + g(\nabla_Y\xi, \phi E_i)\nabla_{E_i}\xi\} = -\nabla_{AY}\xi + \nabla_{AY}\xi = 0.$$

If we use the formula $c\sum_i(\nabla_{E_i}A)E_i = c(\xi\alpha)\xi$ and $cg((\nabla_\xi A)Y, \xi) = c(\xi\alpha)\cdot\eta(Y)$ which come from (2.8), then we get

$$c\sum_i g(\nabla_{E_i}\nabla_Y\xi - \nabla_{\nabla_{E_i}Y}\xi, \phi E_i) = c\{\sum_i g(Y, (\nabla_{E_i}A)E_i) - g((\nabla_\xi A)Y, \xi)\} = 0.$$

Also using the formula $c\sum_i(\nabla_{E_i}A)E_i = c(\xi\alpha)\xi$, we have

$$c\sum_i \nabla_{E_i}\nabla_{\phi E_i}\xi = c\sum_i \{(\nabla_{E_i}A)E_i - g(\xi, (\nabla_{E_i}A)E_i)\xi\} = 0.$$

Thus from these equations we see that (2.9) reduces to the following

$$(2.10)\quad \begin{aligned} \sum_{i,j} cg(\phi E_i, E_j)(\nabla_{E_i}\nabla_{E_j}A)Y &= \frac{c^2}{4}\sum_i \{g(\phi Y, \phi E_i)\nabla_{E_i}\xi - g(\nabla_{E_i}\xi, Y)\phi^2 E_i\} \\ &= \frac{c^2}{4}\{\nabla_Y\xi + \sum_i g(\nabla_{E_i}\xi, Y)E_i\} = \frac{c^2}{2}\phi AY, \end{aligned}$$

where in the last equality we have used the assumption $A\phi + \phi A = 0$.

If we use the Ricci-formula to (2.10) for the shape operator A, then we get

$$(2.11)\quad c\sum_{i,j} g(\phi E_i, E_j)\{R(E_i, E_j)AY - A(R(E_i, E_j)Y)\} = c^2\phi AY.$$

On the other hand, from the equations of Gauss and the assumption $A\phi+\phi A=0$ it follows that

$$\sum_{i,j} g(\phi E_i, E_j)R(E_i, E_j)Y=\frac{c}{4}\sum_i \{g(\phi E_i, Y)E_i-g(E_i, Y)\phi E_i + g(\phi^2 E_i, Y)\phi E_i-g(\phi E_i, Y)\phi^2 E_i-2g(\phi E_i, \phi E_i)\phi Y\} + \sum_i \{g(A\phi E_i, Y)AE_i-g(AE_i, Y)A\phi E_i\}=-cn\phi Y+2A^2\phi Y,$$

from which together with (2.4), (2.11) reduces to

$$c^2\phi AY=0.$$

If $c\neq 0$, then $\phi AY=0$. It follows from the almost contact structure that we have $AY=\alpha\eta(Y)\xi$. The rank of A at a point x in M is called the type number and is denoted by $t(x)$. Thus it means that the type number $t(x)$ of any point x in M is at most 1. It is however seen that (cf. Yano and Kon [13]) $t(x)>1$ at some point x of M for $c\neq 0$. So it is contradiction. Hence we have $c=0$. This completes the above proof.

From Lemma 2.1. we have the following.

Proposition 2.2. *Let M be a real hypersurface of a complex space form $M_n(c)$. If $\phi A+A\phi=0$, then M is cylinderical.*

Proof. From the assumption it follows that $A\xi=\alpha\xi$. Since $c=0$ by Lemma 2.1, (2.3) implies $A\phi A=0$, from which it follows that $(\phi A)^2=\phi A\phi A=-\phi AA\phi=\phi A^t(\phi A)$. Thus $\mathrm{tr}(\phi A)^t(\phi A)=0$, that is, $\phi A=0$. Then $AX=\alpha\eta(X)\xi$. Hence M is cylinderical.

Also by using Lemma 2.1 we get the following.

Lemma 2.3. *Let M be a real hypersurface of a complex space form $M_n(c)$, $c\neq 0$. If ξ is an eigenvector of A, then $\alpha=\eta(A\xi)$ is locally constant.*

Proof. Since $X\alpha=\beta\eta(X)$, we have $\nabla_X \operatorname{grad}\alpha=(X\beta)\xi+\beta\nabla_X\xi$, where we have put $\beta=\xi\alpha$. From which together with the fact $g(\nabla_X \operatorname{grad}\alpha, Y)=g(\nabla_Y \operatorname{grad}\alpha, X)$ it follows that

$$(X\beta)\eta(Y)-(Y\beta)\eta(X)+\beta g((\phi A+A\phi)X, Y)=0. \tag{2.12}$$

Putting $X=\xi$ or $Y=\xi$ in (2.12), we get $X\beta=(\xi\beta)\eta(X)$ or $Y\beta=(\xi\beta)\eta(Y)$. Thus (2.12) reduces to

$$\beta g((\phi A + A\phi)X, Y) = 0.$$

By Lemma 2.1 there are no points on M at which $\phi A + A\phi = 0$, which yields that $\beta = 0$ on M. This means α is constant on M.

Remark. For a real hypersurface of a complex projective space CP^n Maeda proved that α is constant ([7]).

3. Real hypersurfaces of CH^n satisfying certain commutative condition. A characterization of the class of hypersurfaces with more than 3 distinct principal curvatures of CP^n is studied by Kimura [4], who proves the following.

Theorem C. *Let M be a real hypersurface of $CP^n (n \geqq 3)$. Then M satisfies $S\phi = \phi S$ if and only if M lies on a tube of radius r over one of the following Kaehler submanifolds;*

(A) *a totally geodesic CP^k, $(1 \leqq k \leqq n-1)$, where $0 < r < \pi/2$,*

(B) *a complex quadric Q^{n-1}, where $0 < r < \pi/4$ and $\cot^2 2r = n-2$,*

(C) *$CP^1 \times CP^{(n-1)/2}$, where $0 < r < \pi/4$, $\cot^2 2r = 1/(n-2)$ and $n (\geqq 5)$ is odd,*

(D) *complex Grassmann $G_{2,5}(C)$, where $0 < r < \pi/4$, $\cot^2 2r = 3/5$ and $n = 9$,*

(E) *Hermitian symmetric space $SO(10)/U(5)$, where $0 < r < \pi/4$, $\cot^2 2r = 5/9$ and $n = 15$.*

This section is devoted to the investigation about certain real hypersurfaces of CH^n under the condition such that the Ricci tensor and the structure tensor are commutative. Now we introduce the following.

Lemma 3.1. *Let M be a real hypersurface of $CH^n (n \geqq 3)$ and $P = A^2 - fA$ such that f is a smooth function on M. If M satisfies the condition*

$$P\phi = \phi P, \tag{3.1}$$

then ξ is a principal vector at each point of M.

For the real hypersurface of $CP^n (n \geqq 3)$ Kimura [4] proved that ξ is principal under the condition (3.1) by using Cecil-Ryan's method in the paper [2]. If we use the same method as used in [4], we can obtain the above Lemma. Thus we omit the proof of the Lemma 3.1.

By the above Lemma and Lemma 2.3 we get the following

Lemma 3.2. *Let M be a real hypersurface of $CH^n(n \geqq 3)$ satisfying $S\phi = \phi S$. Then the principal curvature α corresponding to ξ is locally constant.*

By Lemma 3.1 we have (2.3). Thus for the complex hyperbolic space CH^n (2.3) implies that $2\phi + 2A\phi A = \alpha(A\phi + \phi A)$. From which, for a unit vector X orthogonal to ξ such that $AX = \lambda X$ we get

$$(2\lambda - \alpha)A\phi X = (\alpha\lambda - 2)\phi X. \tag{3.3}$$

Let V be an open set consisting of points x of M at which $(2\lambda - \alpha)_x \neq 0$. Then $A\phi X = \mu\phi X$ on V, where we have put $\mu = (\alpha\lambda - 2)/(2\lambda - \alpha)$. From (1.5) and (3.1) it follows that

$$(\mu - \lambda)\{(\mu + \lambda) - h\} = 0, \tag{3.4}$$

where h means the trace of A. Thus $\mu = \lambda$ or $h = \lambda + \mu$ holds on V.

For the case $\mu = \lambda$, it is a root of a quadratic equation $x^2 - \alpha x + 1 = 0$ with constant coefficients, which means that λ is constant. Since $\alpha^2 \geqq 4$, we put $\alpha = \pm 2$ or $\alpha = 2 \coth 2\theta$. Then it is seen that $\lambda = \pm 1$ for $\alpha = \pm 2$ and $\lambda = \coth\theta$ or $\tanh\theta$ for $\alpha = 2\coth 2\theta$. This means that we have at most of five kinds of principal curvatures α, $\coth\theta$, $\tanh\theta$, and λ, μ such that $\lambda + \mu = h$. Since (3.3) implies that multiplicities of λ and μ are equal, say m_1, we can put

$$h = (\lambda + \mu)m_1 + m_2 \coth\theta + m_3 \tanh\theta + \alpha,$$

from which together with $h = \lambda + \mu$ it follows that

$$(1 - m_1)h = m_2 \coth\theta + m_3 \tanh\theta + \alpha. \tag{3.5}$$

Since the right hand side of (3.5) is positive or negative according as $\theta > 0$ or $\theta < 0$, respectively, we have $m_1 \neq 1$, and h is constant.

On the other hand, it follows from $h = \lambda + \mu$ that $2\lambda^2 - 2h\lambda + \alpha h - 2 = 0$. Thus all principal curvatures are constant on V. Since V is open and the constancy of principal curvatures gives that V is closed, V coincides with M itself or it is empty. If V is empty, then $2\lambda = \alpha$ gives $\alpha\lambda = 2$ because of (3.3). Thus $\lambda = \pm 1$. Together with this fact we conclude that all principal curvatures are constant on M. Thus we have the following.

Theorem 3.3. *Let M be a real hypersurface of $CH^n(n \geqq 3)$. Then the Ricci tensor of M commutes with the almost contact structure of M induced from CH^n if and only if M is of type A_1, A_2.*

Proof. By the classification Theorem of Berndt M is of type A_1, A_2 or B.

On the other hand, Montiel and Romero show that the real hypersurface M of CH^n is of type A_1, A_2 if and only if the almost contact structure tensor commutes with the second fundamental form. Hence the type A_1, A_2 naturally satisfy $S\phi = \phi S$.

Now we suppose that M is of B-type. Then the table of Berndt [1] gives that $\alpha = 2 \tanh 2\theta$, $\lambda = \tanh \theta$ and $\mu = \coth \theta$. Since multiplicities of λ and μ are equal, (3.4) gives $h = \alpha + (n-1)(\lambda + \mu) = \alpha + (n-1)h$. Thus $(n-2)h + \alpha = 0$, from which together with the fact that $\alpha = 4\lambda/(1+\lambda^2)$ it follows that $4\lambda^2 + (n-2)(1+\lambda^2)^2 = 0$. This contradicts.

Remark. For the real hypersurface of $CP^n (n \geqq 3)$ Kimura [4] proved that $S\phi = \phi S$ if and only if M is of type A_1, A_2, or M is locally congruent to one of a certain hypersurface of type B, C, D or E.

4. Real hypersurfaces of $M_n(c)$, $c \neq 0$. Let M be a pseudo-Einstein real hypersurface of a complex space form $M_n(c)$, $c \neq 0$. Then the Ricci tensor S of M is given by $SX = aX + b\eta(X)\xi$ where a and b are C^∞-functions. From which it naturally satisfies the following.

$$A\xi = \alpha\xi, \tag{4.1}$$

$$R(X, Y)(SZ) + R(Y, Z)(SX) + R(Z, X)(SY) = 0, \tag{4.2}$$

for any X, Y, and Z in $\xi^\perp$, where we have put $\xi^\perp$ the orthogonal complement of ξ in $T_x(M)$ for any x in M.

In this section, we are concerned with the converse problem. Namely we will give another characterization of pseudo-Einstein real hypersurfaces of $M_n(c)$, $c \neq 0$, with (4.1) and (4.2). From (4.1) it follows that $\eta(AX) = 0$ for any X in $\xi^\perp$. By taking account of (1.3) and (1.5), the above equation (4.2) is equivalent to

$$\begin{aligned} &g(QZ, Y)\phi X + g(QX, Z)\phi Y + g(QY, X)\phi Z \\ &+ 2g(\phi Y, Z)\phi PX + 2g(\phi Z, X)\phi PY + 2g(\phi X, Y)\phi PZ = 0 \end{aligned} \tag{4.3}$$

for any X, Y and Z in $\xi^\perp$, where we have put $P = A^2 - hA$, $h = \mathrm{tr}A$, and $Q = P\phi + \phi P$. Since ϕ is non-degenerate on $\xi^\perp$, (4.3) reduces to

$$\begin{aligned} &g(QZ, Y)X + g(QX, Z)Y + g(QX, Y)Z \\ &+ 2\{g(\phi Y, Z)PX + g(\phi Z, X)PY + g(\phi X, Y)PZ\} = 0. \end{aligned} \tag{4.4}$$

For a symmetric transformation $P = A^2 - hA$ let X, Y, and Z be orthonormal eigenvectors such that

$$PX = \alpha_r X,\ PY = \alpha_s Y,\ \text{and}\ PZ = \alpha_t Z. \tag{4.5}$$

Thus, from which together with (4.4) it follows that

$$\begin{aligned} g(QZ, Y) - 2\alpha_r g(\phi Z, Y) &= 0, \\ g(QX, Z) - 2\alpha_s g(\phi X, Z) &= 0, \\ g(QY, X) - 2\alpha_t g(\phi Y, Z) &= 0. \end{aligned} \tag{4.6}$$

Using (4.5) again to (4.6), we have

$$\begin{aligned} (\alpha_s + \alpha_t - 2\alpha_r) g(\phi Y, Z) &= 0, \\ (\alpha_r + \alpha_t - 2\alpha_s) g(\phi X, Z) &= 0, \\ (\alpha_r + \alpha_s - 2\alpha_t) g(\phi Y, X) &= 0. \end{aligned} \tag{4.7}$$

Let us now decompose $T_x(M)$ as following : $T_x(M) = P(\alpha_1) \oplus \cdots \oplus P(\alpha_p)$, where $P(\alpha_r) = \{X \in T_x(M) \mid PX = \alpha_r X\}$ $(r = 1, \ldots, p)$, $\alpha_1, \ldots, \alpha_p$ are all distinct, and ξ in $P(\alpha_1)$.

Lemma 4.1. *If* $p \geqq 2$ *and* $\dim P(\alpha_1) \geqq 2$, *then* $\dim P(\alpha_1) = 2$, *and* $\dim P(\alpha_r) = 1 (r \geqq 2)$.

Proof. Suppose $\dim P(\alpha_1) \geqq 3$ or $\dim P(\alpha_r) \geqq 2$ for some $r \geqq 2$. Then for any $s \leqq p$, $s \neq r$, and any linearly independent vectors X, Y in $P(\alpha_r) (r = 1, \ldots, p)$, and Z in $P(\alpha_s)$, (4.7) give rise to

$$\begin{aligned} (\alpha_s - \alpha_r) g(\phi Y, Z) &= 0, \\ (\alpha_s - \alpha_r) g(\phi X, Z) &= 0, \\ (\alpha_r - \alpha_s) g(\phi Y, X) &= 0. \end{aligned} \tag{4.8}$$

Since $\alpha_r \neq \alpha_s$, $g(\phi Y, Z) = g(\phi X, Z) = g(\phi Y, X) = 0$, from which it follows that ϕX is orthogonal to $P(\alpha_s)$ for any s different from r. Thus ϕX is contained in $P(\alpha_r)$. In particular, if we put $Y = X$, then $g(\phi X, Y) = g(\phi X, \phi X) \neq 0$. This contradicts. Thus we have the above Lemma.

Lemma 4.2. *If* $p \geqq 2$, *then* $\dim P(\alpha_1) = 1$.

Proof. If we suppose $\dim P(\alpha_1) \neq 1$, then by Lemma 1, we get dim $P(\alpha_1) = 2$. Thus we can take a vector X in $P(\alpha_1)$ orthogonal to ξ. Since $\dim P(\alpha_1) = 2$, ϕX is not contained in $P(\alpha_1)$. Thus ϕX is in $P(\alpha_2) \oplus \cdots \oplus P(\alpha_p)$. Hence we can assume that there exists an element Y in $P(\alpha_2)$ such that $g(\phi X, Y) \neq 0$.

Now let $P \geqq 3$. Then let us take X, Y, and Z be orthonormal vectors in $P(\alpha_1)$, $P(\alpha_2)$, and $P(\alpha_r)$ $(r \geqq 3)$, respectively. From which and (4.7) it follows that $(\alpha_1+\alpha_2-2\alpha_r)g(\phi X, Y) = 0$. Thus, we get $2\alpha_r = \alpha_1+\alpha_2$ for $r \geqq 3$ because of $g(\phi X, Y) \neq 0$. Hence we have $p = 3$. This implies that $\dim P(\alpha_1) + \dim P(\alpha_2) + \dim P(\alpha_3) = 4$ by virtur of Lemma 1. This contradicts the fact $\dim T_x(M) \geqq 5$ for $n \geqq 3$. Thus we should have $p = 2$. But in this case we also have $\dim P(\alpha_1) + \dim P(\alpha_2) = 3$ by Lemma 1. This also makes contradiction. Thus we get the above Lemma.

Lemma 4.3. $p = 2$.

Proof. Firstly we now consider for the case $p \geqq 3$. Then by Lemma 4.2. $\dim P(\alpha_1) = 1$. And we will show $\dim P(\alpha_r) = 1$ $(r \geqq 2)$ for $p \geqq 3$. Thus, if we suppose $\dim P(\alpha_r) \geqq 2$ for some $r \geqq 2$, then for any linearly independent vectors X, Y in $P(\alpha_r)$ and Z in $P(\alpha_s)$, $r \neq s$, $s \geqq 2$, we get $g(\phi X, Y) = g(\phi Y, Z) = g(\phi Z, X) = 0$ by virture of (4.8). Hence we evoke the same contradiction as Lemma 4.1. Thus we have $\dim P(\alpha_r) = 1$ for any $r \geqq 2$.

Now we consider for $p \geqq 4$. Then from above facts $\dim P(\alpha_r) = 1$ for any $r \geqq 2$. Thus for X in $P(\alpha_2)$, ϕX is contained in $P(\alpha_3) \oplus \cdots \oplus P(\alpha_p)$. Hence we can take an element Y in $P(\alpha_3)$ such that $g(\phi X, Y) \neq 0$. For Z in $P(\alpha_r)$, $r \geqq 4$, we have

$$(\alpha_2+\alpha_3-2\alpha_r)g(\phi X, Y) = 0.$$

Since $g(\phi X, Y) \neq 0$, we get $2\alpha_r = \alpha_2+\alpha_3$ for $r \geqq 4$. Thus $p = 4$. This implies $\sum_{r=1}^{4} \dim P(\alpha_r) = 4$. This contradicts. Hence $p = 3$. For this case we can also have $\sum_{r=1}^{3} \dim P(\alpha_r) = 3$. This also makes contradiction. Thus we should have $p = 2$.

From Lemmas 4.1, 4.2 and 4.3 we get the following.

Theorem 4.4. *Let M be a real hypersurface of a complex space form $M_n(c)$, $c \neq 0$. If M satisfies* (4.1) *and* (4.2), *then M is pseudo-Einstein.*

Proof. By Lemma 4.3 we have $\dim P(\alpha_1) = 1$, and $\dim P(\alpha_2) = 2n-2$. Thus

$$P=\begin{pmatrix} \alpha_1 & & & 0 \\ & \alpha_2 & & \\ & & \ddots & \\ 0 & & & \alpha_2 \end{pmatrix}$$

This gives $P=\alpha_2 I+(\alpha_1-\alpha_2)\eta\otimes\xi$. From which and (1.5) it follows that $S=\{(2n+1)c/4-\alpha_2\}I+(\alpha_2-\alpha_1-3c)\eta\otimes\xi$. Hence M is pseudo-Einstein.

Remark. Recently Kimura and Maeda [5] introduced the notion of η-parallel second fundamental form A, that is, $g((\nabla_X A)Y, Z)=0$ for any X, Y, and Z in $\xi^\perp$. And they showed that any real hypersurface M of CP^n with η-parallel second fundamental form A and principal vector ξ is of type A_1, A_2 and B.

The condition ξ is principal can not be omitted because a ruled real hypersurface M in CP^n has η-parallel second fundamental form A but ξ is not principal.

5. Real hypersurfaces of CP^n satisfying certain conditions. To give another characterization of some type of real hypersurfaces of the complex projective space CP^n we now introduce the following.

Lemma 5.1. (Takagi [12]) *If M is a connected complete totally η-umbilical real hypersurface in $CP^n(n\geq 2)$, then M is of type A_1.*

Lemma 5.2. (Yano and Kon [14]) *Let M be a connected complete real hypersurface in $CP^n(n\geq 3)$. If $\phi A+A\phi=k\phi$ for some constant $k\neq 0$, then M is of type A_1 or B.*

By above Lemmas we can see that the type A_1 or B satisfies the condition

$$(*)\qquad S\phi+\phi S=k_1\phi\quad (k_1:\text{ constant}).$$

And also pseudo-Einstein real hypersurfaces of CP^n satisfy (*). As the converse problem in this section we are devoted to the investigation of the real hypersurfaces of CP^n satisfying (*) and with principal structure vector field ξ.

By (1.5), (*) is equivalent to

$$(5.1)\qquad A^2\phi+\phi A^2-h(A\phi+\phi A)=k\phi,$$

where we have put $k=2(2n+1)-k_1$, and h means the trace of A.

Since CP^n has the Fubini-Study metric and the constant holomorphic sectional curvature $c = 4$, (2.3) implies that

$$\alpha(\phi A + A\phi) - 2A\phi A + 2\phi = 0. \tag{5.2}$$

From which it follows that if X is an eigenvector of A with eigenvalue λ and if X is orthogonal to ξ, then ϕX is an eigenvector of A with eigenvalue $\mu = (\alpha\lambda + 2)/(2\lambda - \alpha)$. With this fact (5.1) implies

$$\mu^2 + \lambda^2 - h(\mu + \lambda) = k. \tag{5.3}$$

Substituting $\mu = (\alpha\lambda + 2)/(2\lambda - \alpha)$ into (5.3), we get

$$\begin{aligned} &4\lambda^4 - 4(\alpha + h)\lambda^3 + 2(\alpha^2 + h\alpha - 2k)\lambda^2 \\ &\quad + 4(\alpha - h + k\alpha)\lambda + 4 + 2\alpha h - \alpha^2 k = 0. \end{aligned} \tag{5.4}$$

Let λ_1, λ_2, λ_3 and λ_4 be the roots of the above equation. Then from the roots and coefficient of (5.4) it follows that

$$\begin{cases} \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = \alpha + h, \\ \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_1\lambda_4 + \lambda_2\lambda_3 + \lambda_2\lambda_4 + \lambda_3\lambda_4 = (\alpha^2 + h\alpha - 2k)/2, \\ \lambda_1\lambda_2\lambda_3 + \lambda_1\lambda_3\lambda_4 + \lambda_2\lambda_3\lambda_4 + \lambda_1\lambda_2\lambda_4 = -(\alpha - h + k\alpha), \\ \lambda_1\lambda_2\lambda_3\lambda_4 = (4 + 2\alpha h - \alpha^2 k)/4. \end{cases} \tag{5.5}$$

Substituting $h = \alpha + m_1\lambda_1 + m_2\lambda_2 + m_1(\alpha\lambda_1 + 2)/(2\lambda_1 - \alpha) + m_2(\alpha\lambda_2 + 2)/(2\lambda_2 - \alpha)$ into the above equation, and noticing α and k are constant, we can see that (5.5) consists of 4 linearly independent equation, where m_j denotes the (constant) multiplicities of pricipal curvatures $(j = 1, 2)$. Thus M has at most five distinct constant principal curvatures. Hence by Theorem A, M is homogeneous. Then by Takagi's classification of homogeneous real hypersurfaces we can suppose M is of type A_1, A_2, B, C, D, and E.

Firstly, we suppose that M is one of type B, C, D and E. Then from the table given in [11] its type has the following roots : $\lambda = \cot(r - \pi/4)$, $\mu = -\tan(r - \pi/4)$, and $\alpha = 2\cot 2r$. Hence $\lambda + \mu = -4/\alpha$, $\lambda\mu = -1$. Thus, by (5.3) we get $k = (4/\alpha)^2 + h(4/\alpha) + 2$. From which, (5.4) can be rewritten as following

$$\begin{aligned} &2\alpha^2\lambda^4 - 2\alpha^2(\alpha + h)\lambda^3 + \{\alpha^4 + h\alpha^3 - 4\alpha^2 - 8h\alpha - 32\}\lambda^2 \\ &\quad + 2(3\alpha^3 + 3h\alpha^2 + 16\alpha)\lambda - \alpha^2(\alpha^2 + \alpha h + 6) = 0 \end{aligned} \tag{5.6}$$

Then (5.6) can be decomposed into

$$(\alpha\lambda^2 + 4\lambda - \alpha)(2\alpha\lambda^2 - 2(\alpha^2 + h\alpha + 4)\lambda + (\alpha^3 + h\alpha^2 + 6\alpha)) = 0. \tag{5.7}$$

Since cot $(r-\pi/4)$, $-\tan(r-\pi/4)$ satisfy $\alpha\lambda^2+4\lambda-\alpha=0$, another roots cot r, $-\tan r$ of C, D, and E should satisfy

$$(5.8)\qquad 2\alpha\lambda^2-2(\alpha^2+h\alpha+4)\lambda+\alpha^3+h\alpha^2+6\alpha=0.$$

On the other hand, cot r, $-\tan r$ are roots of $\lambda^2-\alpha\lambda-1=0$. From this fact and the root and coefficient of of (5.8), it follows that

$$h\alpha+4=0, \text{ and } \alpha^2+h\alpha+8=0.$$

Thus $\alpha^2+4=0$. This contradicts. Thus the type of C, D, and E can not occur.

Next we consider for the type A_1, A_2. Then we introduce the following.

Lemma 5.3. (Okumura [10]) *Let M be a real hypersurface of CP^n. Then M is of type A_1 or A_2 if and only if $A\phi=\phi A$.*

Since by Lemma 5.1. the type A_1 naturally satisfies (*) and its structure vector ξ is principal, we restrict our attention to the type A_2. Then using Lemma 5.3 to (5.1), we get

$$(5.9)\qquad A^2\phi-hA\phi=k\phi/2.$$

From the table of type A_2 given in [11] it follows that for an eigenvector X such that $AX=-\tan r\, X$

$$(5.10)\qquad 2\cot^2 r-2h\cot r=k.$$

Also for the case $AX=\cot rX$, $A\phi X=-\tan r\phi X$ we get

$$(5.11)\qquad 2\tan^2 r+2h\tan r=k.$$

From (5.10) and (5.11) it follows that $(\cot r+\tan r)(\cot r-\tan r-h)=0$. Since $\cot r+\tan r\neq 0$, $h=\cot r-\tan r=\alpha$. Thus $k=2$. Then (*) implies $S\phi+\phi S=4n\phi$. From which and Lemma 5.3, it follows $S\phi=\phi S=2n\phi$. Hence $S=2nI-2\eta\otimes\xi$. Thus the type of A_2 satisfying (*) is pseudo-Einstein and M is $M(2n-1, m,(m-1)/(n-m))$ (cf. Yano and Kon [13]). Hence we have the following.

Theorem 5.4. *Let M be a connected complete real hypersurface of CP^n and assume that ξ is principal vector field on M. If M satisfies (*), then M is of type A_1, B or M is locally congruent to one of a certain hypersurface of type A_2.*

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(Received March 27, 1989)