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QUASI KO_*-EQUIVALENCES; WOOD SPECTRA AND ANDERSON SPECTRA

Dedicated to Professor Yukihiro Kodama on his 60th birthday

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Let KO, KU and KC denote the real, complex and self-conjugate K-spectrum respectively. Given CW-spectra X, Y we say that X is *quasi KO_*-equivalent to* Y if there exists a map $h: Y \to KO \wedge X$ such that the composite $(\mu \wedge 1)(1 \wedge h): KO \wedge Y \to KO \wedge X$ is an equivalence where $\mu: KO \wedge KO \to KO$ denotes the multiplication of KO (see [Y2]). The KU-homology KU_*X is regarded as a $Z/2$-graded abelian group with involution, since the conjugation $t_u: KU \to KU$ gives an involution t_{u*} on KU_*X for any CW-spectrum X. Notice that KU_*X and KU_*Y are isomorphic as $Z/2$-graded abelian groups with involution if X is quasi KO_*-equivalent to Y.

Let us denote by P and Q the cofibers of the maps $\eta: \Sigma^1 \to \Sigma^0$ and $\eta^2: \Sigma^2 \to \Sigma^0$ respectively where $\eta: \Sigma^1 \to \Sigma^0$ denotes the stable Hopf map of order 2. As is well known, $KU_0P \cong Z \oplus Z$ on which $t_{u*} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $KU_1P = 0$, and $KU_0Q \cong Z \cong KU_{-1}Q$ on both of which $t_{u*} = 1$. Following [MOY] we call a CW-spectrum X *a Wood spectrum* if X is quasi KO_*-equivalent to P, and *an Anderson spectrum* if X is quasi KO_*-equivalent to Q (see [Y2]). For any abelian group G we denote by SG the Moore spectrum of type G. Evidently $KU_0SG \cong G$ on which $t_{u*} = 1$ and $KU_1SG = 0$.

Let X be a CW-spectrum such that

i) KU_*X is pure projective and 2-torsion free, thus it is a direct sum of a free group and cyclic p-groups ($p \neq 2$), or

ii) KU_*X is pure injective and 2-divisible, thus it is a direct summand of a direct product of a divisible group and cyclic p-groups ($p \neq 2$) (see [F]).

Then KU_*X admits a direct sum decomposition $KU_*X \cong A \oplus B \oplus C \oplus C$ so that the conjugation t_{u*} on KU_*X behaves as

$$t_{u*} = 1 \text{ on } A, \ t_{u*} = -1 \text{ on } B \text{ and } t_{u*} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ on } C \oplus C$$

respectively (use [B, Propositions 3.7 and 3.8] or [CR]).

In [Y2, Theorems 1 and 2] (cf. [MOY]) we have obtained certain results concerning Wood spectra and Anderson spectra. On the other hand, Bousfield [B, Theorems 3.2 and 3.3] has independently shown the following complete results which contain our partial results, although Bousfield's notation (or statement) is different from ours.

Theorem 1 (Bousfield). *Let X be a CW-spectrum such that KU_*X is pure projective and 2-torsion free. Then there exist abelian groups A_i $(0 \leqq i \leqq 7)$, C_j $(0 \leqq j \leqq 1)$ and G_k $(0 \leqq k \leqq 3)$ with C_j and G_k free so that X is quasi KO_*-equivalent to the wedge sum* $(\bigvee_i \Sigma^i SA_i) \vee (\bigvee_j \Sigma^j P \wedge SC_j) \vee (\bigvee_k \Sigma^{k+1} Q \wedge SG_k)$. (Theorem 2.4).

Theorem 2 (Bousfield). *Let X be a CW-spectrum such that KU_*X is pure injective and 2-divisible. Then there exist abelian groups A_i $(0 \leqq i \leqq 7)$, C_j $(0 \leqq j \leqq 1)$ and G_k $(0 \leqq k \leqq 3)$ with C_j and G_k divisible 2-torsion so that X is quasi KO_*-equivalent to the wedge sum* $(\bigvee_i \Sigma^i SA_i) \vee (\bigvee_j \Sigma^j P \wedge SC_j) \vee (\bigvee_k \Sigma^{k+1} Q \wedge SG_k)$. (Theorem 3.4).

Strictly speaking, Bousfield has proved that an associative KO-module spectrum W is isomorphic as KO-module spectra to an extended KO-module spectrum $KO \wedge Y$ with $Y = (\bigvee_i \Sigma^i SA_i) \vee (\bigvee_j \Sigma^j P \wedge SC_j) \vee (\bigvee_k \Sigma^{k+1} Q \wedge SG_k)$, if $\pi_*(W \wedge P)$ is free or divisible. Our purpose in this note is to give a new proof of Theorems 1 and 2 by applying our method developed in [Y2, Y3]. Our method allows us to prove Bousfield's result for any associative KO-module spectrum W, although we here give a new proof of his result only for an extended KO-module spectrum $KO \wedge X$.

In §1 we recall some properties of K-spectra KO, KU and KC ([B] or [An1]) and then study the structure of KC_*X for any CW-spectrum X as in Theorem 1 or 2. In §2 and §3 we will only deal with CW-spectra X as in Theorems 1 and 2 respectively. After giving a refined decomposition of KU_*X in each case, we will prove Theorem 1 (Theorem 2.4) along the line adopted in [Y2, Y3] and Theorem 2 (Theorem 3.4) by a dual argument. In the proof of Theorem 2 we use the Anderson universal coefficient sequences (see [An2] or [Y1]), as was implicitly suggested in [B].

In this note we will work in the stable homotopy category of CW-spectra [Ad].

1. The real, complex and self-conjugate K-spectrum.

1.1. Let KO, KU and KC denote the real, complex and self-conjugate K-spectrum respectively. All of these K-spectra are associative and commutative ring spectra with unit. As relations among these K-spectra we have the following cofiber sequences ([An1], [B]):

(1.1) i) $\Sigma^1 KO \xrightarrow{\eta\wedge 1} KO \xrightarrow{\varepsilon_u} KU \xrightarrow{\varepsilon_o\pi_u^{-1}} \Sigma^2 KO$

ii) $\Sigma^2 KO \xrightarrow{\eta^2\wedge 1} KO \xrightarrow{\varepsilon_c} KC \xrightarrow{\tau\pi_c^{-1}} \Sigma^3 KO$

iii) $KC \xrightarrow{\zeta} KU \xrightarrow{\pi_u^{-1}(1-t_u)} \Sigma^2 KU \xrightarrow{\gamma\pi_u} \Sigma^1 KC$

iv) $\Sigma^1 KC \xrightarrow{(-\tau,\ \tau\pi_c^{-1})} KO \vee \Sigma^4 KO \xrightarrow{\varepsilon_u \vee \pi_u^2\varepsilon_u} KU \xrightarrow{\varepsilon_c\varepsilon_o\pi_u^{-1}} \Sigma^2 KC$

v) $\Sigma^2 KU \xrightarrow{(-\varepsilon_o\pi_u,\ \varepsilon_o\pi_u^{-1})} KO \vee \Sigma^4 KO \xrightarrow{\varepsilon_c \vee \pi_c\varepsilon_c} KC \xrightarrow{\varepsilon_u\tau\pi_c^{-1}} \Sigma^3 KU.$

The maps involved in (1.1) admit several properties as follows. The stable Hopf map $\eta: \Sigma^1 \to \Sigma^0$ has order 2. The maps $\varepsilon_u : KO \to KU$, $\varepsilon_c : KO \to KC$ and $\zeta : KC \to KU$ are ring maps with $\zeta\varepsilon_c = \varepsilon_u$, and the maps $\varepsilon_o : KU \to KO$, $\tau : \Sigma^1 KC \to KO$ and $\gamma : KU \to \Sigma^1 KC$ are merely KO-module maps with $\tau\gamma = \varepsilon_o$. The periodicity maps $\pi_u : \Sigma^2 KU \to KU$ and $\pi_c : \Sigma^4 KC \to KC$ satisfy $\zeta\pi_c = \pi_u^2\zeta$ and $\pi_c\gamma = \gamma\pi_u^2$ respectively. The conjugation maps $t_u : KU \to KU$ and $t_c : KC \to KC$ are ring maps satisfying $t_u^2 = 1$, $t_c^2 = 1$, $t_u\pi_u = -\pi_u t_u$ and $t_c\pi_c = \pi_c t_c$, and besides

(1.2) $t_c\varepsilon_c = \varepsilon_c$, $\tau t_c = -\tau$, $t_u\zeta = \zeta t_c = \zeta$ *and* $t_c\gamma = -\gamma t_u = -\gamma$.

Moreover there hold the following equalities among these maps (see [B, 1.9]):

(1.3) i) $\varepsilon_o\varepsilon_u = 2$, $\tau\varepsilon_c = \eta\wedge 1$, $\tau\pi_c\varepsilon_c = 0$, $\pi_c\varepsilon_c\tau\pi_c^{-1} = \varepsilon_c\tau + \eta\wedge 1$, $\zeta\gamma = 0$ *and* $\gamma\pi_u\zeta = \eta\wedge 1$, *and also*

ii) $\varepsilon_u\varepsilon_o = 1 + t_u$, $\gamma\varepsilon_u\tau = 1 - t_c$ *and* $\varepsilon_c\varepsilon_o\zeta = 1 + t_c$.

Let K denote the K-spectrum KO, KU or KC. To any map $f: Y \to K\wedge X$ we assign a K-module map $\kappa_k(f) = (\mu\wedge 1)(1\wedge f) : K\wedge Y \to K\wedge X$ where $\mu: K\wedge K \to K$ denotes the multiplication of K. The assignment $\kappa_k : [Y, K\wedge X] \to [K\wedge Y, K\wedge X]$ gives a right inverse of the induced homomorphism $(\iota\wedge 1)^* : [K\wedge Y, K\wedge X] \to [Y, K\wedge X]$ where $\iota : \Sigma^0 \to K$ denotes the unit of K. This homomorphism κ_k induces a homomorphism

(1.4) $\kappa_\iota^k : [Y, K\wedge X] \to \mathrm{Hom}(K_\iota Y, K_\iota X)$

assigning any map f to its induced homomorphism $\varkappa_k(f)_*$ in dimension i, which is often abbreviated as $\varkappa_i$.

Let $\nabla K(G)$ denote the Anderson dual spectrum of K with coefficients in G (see [An2] or [Y1, I and II]). The CW-spectra K and $\nabla K(G)$ are related by the following universal coefficient sequence

$$0 \to \mathrm{Ext}(K_{*-1}X, G) \to \nabla K(G)^*X \to \mathrm{Hom}(K_*X, G) \to 0.$$

Recall that $\nabla KU(G) \cong KU \wedge SG$, $\nabla KO(G) \cong \Sigma^{-4} KO \wedge SG$ and $\nabla KC(G) \cong \Sigma^{-3} KC \wedge SG$ where SG denotes the Moore spectrum of type G ([An2] or [Y1, I]). So we may rewrite the above universal coefficient sequence as follows:

(1.5) i) $0 \to \mathrm{Ext}(KU_{-1}X, G) \to [X, KU \wedge SG] \xrightarrow{\varkappa_0^{ku}} \mathrm{Hom}(KU_0X, G) \to 0$

ii) $0 \to \mathrm{Ext}(KO_3X, G) \to [X, KO \wedge SG] \xrightarrow{\varkappa_4^{ko}} \mathrm{Hom}(KO_4X, G) \to 0$

iii) $0 \to \mathrm{Ext}(KC_2X, G) \to [X, KC \wedge SG] \xrightarrow{\varkappa_3^{ko}} \mathrm{Hom}(KC_3X, G) \to 0.$

1.2. In this note we will only deal with a CW-spectrum X such that

(1.6) i) KU_*X is pure projective and 2-torsion free, thus it is written as a direct sum of a free group and cyclic p-groups ($p \neq 2$), or

ii) KU_*X is pure injective and 2-divisible, thus it is written as a direct summand of a direct product of a divisible group and cyclic p-groups ($p \neq 2$) (see [F]).

Given such a CW-spectrum X, KU_0X and KU_1X are respectively decomposed into the forms of

(1.7) $KU_0X \cong A \oplus B \oplus C \oplus C$ and $KU_1X \cong D \oplus E \oplus F \oplus F$

on which the conjugation t_{u*} behaves as follows:

(1.8) $t_{u*} = 1$ on A or D, $t_{u*} = -1$ on B or E, and
$t_{u*} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ on $C \oplus C$ or $F \oplus F$.

Here C and F may be taken to be free in the (1.6) i) case, and to be divisible 2-torsion in the (1.6) ii) case (see [B, Propositions 3.7 and 3.8] or [CR]).

In order to compute KC_*X we use the short exact sequence

$$0 \to (\gamma\pi_u)_*(KU_{i-1}X) \to KC_iX \to \zeta_*(KC_iX) \to 0$$

induced by the cofiber sequence (1.1) iii) (see [Y2, Lemma 2.1 i)]). Since the composite homomorphism $(\zeta\varepsilon_c\varepsilon_o)_* : KU_iX \to KU_iX$ restricted to the image $\zeta_*(KC_iX)$ is just multiplication by 2, the above short exact sequence is split after tensored with $Z\left[\frac{1}{2}\right]$. Under our assumption (1.6) it is a pure exact sequence, and actually a split exact sequence. Thus KC_*X admits the following direct sum decomposition:

$$(1.9)\quad \begin{aligned} KC_0X &\cong (A \oplus B * Z/2 \oplus C) \oplus (D \oplus E \otimes Z/2 \oplus F),\\ KC_1X &\cong (D \oplus E * Z/2 \oplus F) \oplus (A \otimes Z/2 \oplus B \oplus C),\\ KC_2X &\cong (A * Z/2 \oplus B \oplus C) \oplus (D \otimes Z/2 \oplus E \oplus F),\\ KC_3X &\cong (D * Z/2 \oplus E \oplus F) \oplus (A \oplus B \otimes Z/2 \oplus C).\end{aligned}$$

Since $\eta \wedge 1 = \gamma\pi_u\zeta : \Sigma^1 KC \to KC$, the induced homomorphisms η_* : $KC_0X \to KC_1X$ restricted to A and $B * Z/2$ are respectively identified with the canonical projection $A \to A \otimes Z/2$ and the canonical inclusion $B * Z/2 \to B$, and the one η_* restricted to the other components $C \oplus (D \oplus E \otimes Z/2 \oplus F)$ is trivial. Thus $\eta_* : KC_0X \cong A \oplus B * Z/2 \oplus C \oplus D \oplus E \otimes Z/2 \oplus F \to KC_1X \cong D \oplus E * Z/2 \oplus F \oplus A \otimes Z/2 \oplus B \oplus C$ is given by

$$(1.10)_0\quad \eta_*(a, b, c, d, [e], f) = (0, 0, 0, [a], b, 0)$$

where [ ] stands for the mod 2 reduction. For $\eta_* : KC_iX \to KC_{i+1}X$, $1 \leqq i \leqq 3$, we can obtain similar expressions $(1.10)_i$ to $(1.10)_0$, which will be used later.

On the other hand, the composite homomorphism $(\gamma\zeta)_* : KC_1X \cong D \oplus E * Z/2 \oplus F \oplus A \otimes Z/2 \oplus B \oplus C \to KC_0X \cong A \oplus B * Z/2 \oplus C \oplus D \oplus E \otimes Z/2 \oplus F$ is given by

$$(1.11)_0\quad (\gamma\zeta)_*(d, e, f, [a], b, c) = (0, 0, 0, d, 0, 2f).$$

For $(\gamma\zeta)_* : KC_{i-1}X \to KC_iX$, $1 \leqq i \leqq 3$, we can also obtain similar expressions $(1.11)_i$ to $(1.11)_0$.

The conjugation t_{c*} on $KC_iX \cong \zeta_*(KC_iX) \oplus (\gamma\pi_u)_*(KU_{i-1}X)$ can be represented by the following matrix

$$(1.12)\quad \begin{pmatrix} 1 & 0 \\ t_i & -1 \end{pmatrix} \ (0 \leqq i \leqq 3)$$

for a certain homomorphism $t_i : \zeta_*(KC_iX) \to (\gamma\pi_u)_*(KU_{i-1}X)$. In particular, take $X = \Sigma^1 Q \wedge SG$ when G is free or divisible. Here Q denotes the cofiber of the square $\eta^2 : \Sigma^2 \to \Sigma^0$. Use the following commutative diagram

$$\begin{array}{ccccc} & & & & \leftarrow 0 \\ KU_{i-2}(Q \wedge SG) & \searrow & & \swarrow & KO_{i-1}(Q \wedge SG) \\ \downarrow & & KC_{i-1}(Q \wedge SG) & & \downarrow \\ KO_{i-4}(Q \wedge SG) & \swarrow & & \searrow & KU_{i-1}(Q \wedge SG) \\ 0 \leftarrow & & & & \end{array}$$

in which the diagonal exact sequences are induced by the cofiber sequences (1.1) ii) and iii). Recall that $KU_{-1}(Q \wedge SG) \cong G \cong KU_0(Q \wedge SG)$ on both of which $t_{u*} = 1$, and besides $KO_{i-1}(Q \wedge SG) \cong G, G, G \otimes Z/2$ or $G * Z/2$ according as $i \equiv 0, 1, 2$ or $3 \bmod 4$. By parallel discussions to [Y2, (2.3)] we then observe that

$$t_{c*} = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix} \text{ on } KC_{-1}(Q \wedge SG) \cong G \oplus G$$

$$(1.13) \quad t_{c*} = 1 \text{ on } KC_0(Q \wedge SG) \cong G \oplus G \otimes Z/2 \text{ and } KC_1(Q \wedge SG) \cong G * Z/2 \oplus G \otimes Z/2$$

$$t_{c*} = -1 \text{ on } KC_2(Q \wedge SG) \cong G * Z/2 \oplus G.$$

1.3. The cofiber sequence $\Sigma^2 \xrightarrow{\eta^2} \Sigma^0 \xrightarrow{i_Q} Q \xrightarrow{j_Q} \Sigma^3$ gives the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & \mathrm{Hom}(KC_{-4}SG, KC_0X) & \to & \mathrm{Hom}(KC_{-1}(Q \wedge SG), KC_0X) & \to & \mathrm{Hom}(KC_{-1}SG, KC_0X) & \to & 0 \\ & & \uparrow{\scriptstyle \chi_0} & & \uparrow{\scriptstyle \chi_0} & & \uparrow{\scriptstyle \chi_0} & & \\ 0 & \longrightarrow & [\Sigma^4 SG, KC \wedge X] & \longrightarrow & [\Sigma^1 Q \wedge SG, KC \wedge X] & \longrightarrow & [\Sigma^1 SG, KC \wedge X] & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \chi_1} & & \downarrow{\scriptstyle \chi_1} & & \downarrow{\scriptstyle \chi_1} & & \\ 0 & \to & \mathrm{Hom}(KC_{-3}SG, KC_1X) & \to & \mathrm{Hom}(KC_0(Q \wedge SG), KC_1X) & \to & \mathrm{Hom}(KC_0SG, KC_1X) & \to & 0 \end{array}$$

in which the vertical arrows χ_i $(i = 0, 1)$ are abbreviated χ_i^{kc} of (1.4). Since $Q \wedge Q \cong Q \vee \Sigma^3 Q$ and $KC \cong KO \wedge Q$, all of the three rows are split short exact sequences. Notice that their splittings are compatible with χ_i $(i = 0, 1)$ because the assignment $\chi_{kc} : [Y, KC \wedge X] \to [KC \wedge Y, KC \wedge X]$ admits the induced homomorphism $(\iota \wedge 1)^*$ as a left inverse. Obviously the right lower arrow χ_1 and the left upper one χ_0 become both epimorphisms, because they are isomorphisms if the abelian group G is free.

We here assume that KU_*X is pure projective and 2-torsion free, and hence KC_*X is written into the form of (1.9). For each map $g : \Sigma^1 Q \wedge$

$SG \to KC \wedge X$ we can choose unique maps $g_0 : \Sigma^1 SG \to KC \wedge X$ and $g_1 : \Sigma^4 SG \to KC \wedge X$ with $g_0 = g(i_Q \wedge 1)$, under the direct sum decomposition $[\Sigma^1 Q \wedge SG, KC \wedge X] \cong [\Sigma^1 SG, KC \wedge X] \oplus [\Sigma^4 SG, KC \wedge X]$. Express the induced homomorphisms $\varkappa_1(g_0) : KC_0 SG \to KC_1 X$ and $\varkappa_0(g_1) : KC_{-4} SG \to KC_0 X$ as $\varkappa_1(g_0) = u+v+w : G \to (D \oplus F) \oplus (A \otimes Z/2 \oplus B \oplus C)$ and $\varkappa_0(g_1) = x+y+z : G \to (A \oplus C) \oplus (D \oplus E \otimes Z/2 \oplus F)$ respectively, where $u : G \to D$, $v : G \to F$, $w : G \to A \otimes Z/2 \oplus B \oplus C$, $x : G \to A$, $y : G \to C$ and $z : G \to D \oplus E \otimes Z/2 \oplus F$.

The induced homomorphism $\varkappa_0(g_0) : KC_{-1} SG \to KC_0 X$ is identified with the composite $(\gamma\zeta)_* \varkappa_1(g_0)$. $KC_0 SG \to KC_0 X$ because $(\gamma\zeta)_* : KC_0 SG \to KC_{-1} SG$ is regarded as the identity on G. On the other hand, the induced homomorphism $\varkappa_1(g_1) : KC_{-3} SG \to KC_1 X$ coincides with the mod 2 reduction of the composite $\eta_* \varkappa_0(g_1) : KC_{-4} SG \to KC_1 X$ because $\eta_* : KC_{-4} SG \to KC_{-3} SG$ is just the canonical projection $G \to G \otimes Z/2$. By means of (1.10) and (1.11) we then observe that

(1.14) i) $\varkappa_0(g_0) : KC_{-1} SG \to KC_0 X$ *is expressed as the sum* $u+2v : G \to D \oplus F \subset (A \oplus C) \oplus (D \oplus E \otimes Z/2 \oplus F)$, *and*

ii) $\varkappa_1(g_1) : KC_{-3} SG \to KC_1 X$ *is expressed as the mod* 2 *reduction* $[x] : G \otimes Z/2 \to A \otimes Z/2 \subset (D \oplus F) \oplus (A \otimes Z/2 \oplus B \oplus C)$.

This result implies that the induced homomorphisms $\varkappa_0(g) : KC_{-1}(Q \wedge SG) \to KC_0 X$ and $\varkappa_1(g) : KC_0(Q \wedge SG) \to KC_1 X$ are respectively represented by the following matrices

$$(1.15) \quad \begin{aligned} &\begin{pmatrix} x+y & 0 \\ z & u+2v \end{pmatrix} : G \oplus G \to (A \oplus C) \oplus (D \oplus E \otimes Z/2 \oplus F) \\ &\begin{pmatrix} u+v & 0 \\ w & [x] \end{pmatrix} : G \oplus (G \otimes Z/2) \to (D \oplus F) \oplus (A \otimes Z/2 \oplus B \oplus C) \end{aligned}$$

where $KC_{-1}(Q \wedge SG) \cong KC_{-4} SG \oplus KC_{-1} SG \cong G \oplus G$ and $KC_0(Q \wedge SG) \cong KC_0 SG \oplus KC_{-3} SG \cong G \oplus (G \otimes Z/2)$.

The abelian group G is now assumed to be free. In this situation the assignment $(\varkappa_0, \varkappa_1) : [\Sigma^1 Q \wedge SG, KC \wedge X] \to \mathrm{Hom}(KC_{-1}(Q \wedge SG), KC_0 X) \oplus \mathrm{Hom}(KC_0(Q \wedge SG), KC_1 X)$ is obviously a monomorphism. As in (1.12) we represent the conjugations t_{C*} on $KC_i X$ $(i = 0, 1)$ by matrices $\begin{pmatrix} 1 & 0 \\ t_i & -1 \end{pmatrix}$ for certain homomorphisms $t_0 : A \oplus C \to D \oplus E \otimes Z/2 \oplus F$ and $t_1 : D \oplus F \to A \otimes Z/2 \oplus B \oplus C$. In particular, (1.13) asserts that $t_0 = 1 : G \to G$ and $t_1 = 0 : G \to G \otimes Z/2$ when $X = \Sigma^1 Q \wedge SG$. Since $\varkappa_i((t_C \wedge 1)g) =$

$t_{C*}\kappa_i(g)t_{C*}$ for any map $g: \Sigma^1 Q\wedge SG \to KC\wedge X$, we can easily check that

(1.16) $(t_C\wedge 1)g = g$ *if and only if* $t_0(x+y) = 2z+u+2v$ *and* $t_1(u+v) = 2w$, *where* $\kappa_0(g) = \begin{pmatrix} x+y & 0 \\ z & u+2v \end{pmatrix}$ *and* $\kappa_1(g) = \begin{pmatrix} u+v & 0 \\ w & [x] \end{pmatrix}$ *as in* (1.15).

1.4. We next consider the following commutative diagram

$$\begin{array}{ccccccccc}
0 \to & \mathrm{Hom}(KC_0X, KC_{-1}SG) & \to & \mathrm{Hom}(KC_0X, KC_{-1}(Q\wedge SG)) & \to & \mathrm{Hom}(KC_0X, KC_{-4}SG) & \to 0 \\
& \uparrow{\scriptstyle \kappa_0} & & \uparrow{\scriptstyle \kappa_0} & & \uparrow{\scriptstyle \kappa_0} & \\
0 \longrightarrow & [X, \Sigma^1 KC\wedge SG] & \longrightarrow & [X, \Sigma^1 KC\wedge Q\wedge SG] & \longrightarrow & [X, \Sigma^4 KC\wedge SG] & \longrightarrow 0 \\
& \downarrow{\scriptstyle \kappa_3} & & \downarrow{\scriptstyle \kappa_3} & & \downarrow{\scriptstyle \kappa_3} & \\
0 \to & \mathrm{Hom}(KC_3X, KC_2SG) & \longrightarrow & \mathrm{Hom}(KC_3X, KC_2(Q\wedge SG)) & \to & \mathrm{Hom}(KC_3X, KC_{-1}SG) & \to 0
\end{array}$$

induced by the cofiber sequence $\Sigma^2 \xrightarrow{\eta^2} \Sigma^0 \xrightarrow{i_Q} Q \xrightarrow{j_Q} \Sigma^3$, where the vertical arrows κ_i $(i = 0, 3)$ are abbreviated κ_i^{KC} of (1.4). All of the three rows are split short exact sequences, and their splittings are compatible with κ_i $(i = 0, 3)$. From (1.5) iii) it follows that the right lower arrow κ_3 and the left upper one κ_0 are both epimorphisms, and in particular they become isomorphisms if the abelian group G is divisible.

Assume that KU_*X is pure injective and 2-divisible, and hence KC_*X is written into the form of (1.9). For each map $g: X \to \Sigma^1 KC\wedge Q\wedge SG$ we can choose unique maps $g_0: X \to \Sigma^4 KC\wedge SG$ and $g_1: X \to \Sigma^1 KC\wedge SG$ with $g_0 = (j_Q\wedge 1)g$, under the direct sum decomposition $[X, \Sigma^1 KC\wedge Q\wedge SG] \cong [X, \Sigma^4 KC\wedge SG] \oplus [X, \Sigma^1 KC\wedge SG]$. Express the induced homomorphisms $\kappa_0(g_1): KC_0X \to KC_{-1}SG$ and $\kappa_3(g_0): KC_3X \to KC_{-1}SG$ as $\kappa_0(g_1) = w+u+v: (A\oplus B* Z/2\oplus C)\oplus(D\oplus F) \to G$ and $\kappa_3(g_0) = z+x+y: (D* Z/2\oplus E\oplus F)\oplus(A\oplus C) \to G$ respectively, where $u: D\to G$, $v: F\to G$, $w: A\oplus B* Z/2\oplus C\to G$, $x: A\to G$, $y: C\to G$ and $z: D* Z/2\oplus E\oplus F\to G$. By a similar argument to (1.14) we obtain

(1.17) i) $\kappa_0(g_0): KC_0X \to KC_{-4}SG$ *is expressed as the sum* $x+2y$: $(A\oplus B* Z/2\oplus C)\oplus(D\oplus F) \to A\oplus C \to G$, *and*

ii) $\kappa_3(g_1): KC_3X \to KC_2SG$ *is expressed as the mod* 2 *restriction* $u: (D* Z/2\oplus E\oplus F)\oplus(A\oplus C) \to D* Z/2 \to G* Z/2$.

This result implies that the induced homomorphisms $\kappa_0(g): KC_0X \to KC_{-1}(Q\wedge SG)$ and $\kappa_3(g): KC_3X \to KC_2(Q\wedge SG)$ are respectively represented by the following matrices

$$(1.18)\quad \begin{aligned} &\begin{pmatrix} x+2y & 0 \\ w & u+v \end{pmatrix} : (A \oplus B * Z/2 \oplus C) \oplus (D \oplus F) \to G \oplus G \\ &\begin{pmatrix} u & 0 \\ z & x+y \end{pmatrix} \quad : (D * Z/2 \oplus E \oplus F) \oplus (A \oplus C) \to (G * Z/2) \oplus G \end{aligned}$$

where $KC_{-1}(Q \wedge SG) \cong KC_{-4}SG \oplus KC_{-1}SG \cong G \oplus G$ and $KC_2(Q \wedge SG) \cong KC_2SG \oplus KC_{-1}SG \cong (G * Z/2) \oplus G$.

The abelian group G is now assumed to be divisible. Then the assignment $(\kappa_0, \kappa_3) : [X, \Sigma^1 KC \wedge Q \wedge SG] \to \mathrm{Hom}(KC_0X, KC_{-1}(Q \wedge SG)) \oplus \mathrm{Hom}(KC_3X, KC_2(Q \wedge SG))$ is obviously a monomorphism. The conjugations t_{c*} on KC_iX $(i = 0, 3)$ are represented by matrices $\begin{pmatrix} 1 & 0 \\ t_i & -1 \end{pmatrix}$ for certain homomorphisms $t_0 : A \oplus B * Z/2 \oplus C \to D \oplus F$ and $t_3 : D * Z/2 \oplus E \oplus F \to A \oplus C$. As a result corresponding to (1.16) we can similarly show that

(1.19) $(t_c \wedge 1)g = g$ *if and only if* $(u+v)t_0 = x+2y-2w$ *and* $(x+y)t_3 = -2z$, *where* $\kappa_0(g) = \begin{pmatrix} x+2y & 0 \\ w & u+v \end{pmatrix}$ *and* $\kappa_3(g) = \begin{pmatrix} u & 0 \\ z & x+y \end{pmatrix}$ *for any map* $g : X \to \Sigma^1 KC \wedge Q \wedge SG$ *as in* (1.18).

1.5. When an abelian group G is free, we consider the following commutative diagram

$$\begin{array}{ccccccccc}
 & & & & \mathrm{Hom}(KU_0(Q \wedge SG), KU_0X) & \xrightarrow{(i_{Q*})^*} & \mathrm{Hom}(KU_0SG, KU_0X) & & \\
 & & & & \uparrow{\scriptstyle \kappa_0} & & \uparrow{\scriptstyle \kappa_0} & & \\
0 & \longrightarrow & [\Sigma^3 SG, KU \wedge X] & \xrightarrow{(j_Q \wedge 1)^*} & [Q \wedge SG, KU \wedge X] & \xrightarrow{(i_Q \wedge 1)^*} & [SG, KU \wedge X] & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \kappa_1} & & \downarrow{\scriptstyle \kappa_1} & & & & \\
 & & \mathrm{Hom}(KU_{-2}SG, KU_1X) & \xrightarrow{(j_{Q*})^*} & \mathrm{Hom}(KU_1(Q \wedge SG), KU_1X) & & & &
\end{array}$$

in which the right vertical arrow $\kappa_0 = \kappa_0^{ku}$ and the left one $\kappa_1 = \kappa_1^{ku}$ are both isomorphisms, and the top and the bottom horizontal arrows $(i_{Q*})^*$ and $(j_{Q*})^*$ are also isomorphisms. The induced homomorphism $(j_Q \wedge 1)^* : [\Sigma^3 SG, KU \wedge X] \to [Q \wedge SG, KU \wedge X]$ admits as a left inverse the composite $((j_{Q*})^* \kappa_1)^{-1} \kappa_1$, which is compatible with the conjugations $(t_u \wedge 1)_*$ because $\kappa_1((t_u \wedge 1)f) = t_{u*} \kappa_1(f) t_{u*}$. In other words, there exists a homomorphism

(1.20) $\lambda : [SG, KU \wedge X] \to [Q \wedge SG, KU \wedge X]$

satisfying $(i_Q \wedge 1)^* \lambda = 1$ and $(t_u \wedge 1)_* \lambda = \lambda (t_u \wedge 1)_*$.

Lemma 1.1. *Let G be a free abelian group and $g': \Sigma^1 Q \wedge SG \to KC \wedge X$ be a map satisfying $(t_c \wedge 1)g' = g'$. If the composite $(\eta \wedge 1)(\tau\pi_c^{-1} \wedge 1)g'(i_Q \wedge 1): SG \to \Sigma^1 KO \wedge X$ is trivial, then there exist maps $h_1: \Sigma^1 SG \to KO \wedge X$ and $g: \Sigma^1 Q \wedge SG \to KC \wedge X$ such that $h_1(j_Q \wedge 1) = (\tau\pi_c^{-1} \wedge 1)g$, $(t_c \wedge 1)g = g$ and $(\zeta \wedge 1)g = (\zeta \wedge 1)g'$* (cf. [Y3, Lemma 1.1]).

Proof. First choose a map $h_0: \Sigma^1 SG \to KO \wedge X$ satisfying $(\varepsilon_u \wedge 1)h_0 = (\zeta \wedge 1)g'(i_Q \wedge 1)$, and then a map $\ell': SG \to KU \wedge X$ such that $(\varepsilon_c \wedge 1)h_0 = g'(i_Q \wedge 1) + (\gamma\pi_u \wedge 1)\ell'$. Composing the conjugation map $t_c \wedge 1$ after the second equality, we see that $2(\gamma\pi_u \wedge 1)\ell' = 0$ because $t_c\varepsilon_c = \varepsilon_c$ and $t_c\gamma = -\gamma$. So there exists a map $k': SG \to KU \wedge X$ satisfying $2\ell' = (1 + t_u \wedge 1)k'$. Applying the right inverse λ of $(i_Q \wedge 1)^*$ obtained in (1.20) onto the above equality, we show that the composite $(\gamma\pi_u \wedge 1)\lambda(\ell'): \Sigma^1 Q \wedge SG \to KC \wedge X$ has order 2. Setting $g = g' + (\gamma\pi_u \wedge 1)\lambda(\ell')$, its map satisfies the equalities $(\varepsilon_c \wedge 1)h_0 = g(i_Q \wedge 1)$, $(t_c \wedge 1)g = g$ and $(\zeta \wedge 1)g = (\zeta \wedge 1)g'$. Using the first equality we can then find a map $h_1: \Sigma^1 SG \to KO \wedge X$ with $h_1(j_Q \wedge 1) = (\tau\pi_c^{-1} \wedge 1)g$.

When an abelian group G is divisible, we next consider the following commutative diagram

$$\begin{array}{ccccccccc}
 & & & & \mathrm{Hom}(KU_1X, KU_1(Q \wedge SG)) & \xrightarrow{j_{Q**}} & \mathrm{Hom}(KU_1X, KU_{-2}SG) & & \\
 & & & & \uparrow{\scriptstyle \kappa_1} & & \uparrow{\scriptstyle \kappa_1} & & \\
0 & \longrightarrow & [X, KU \wedge SG] & \xrightarrow{(1 \wedge i_Q \wedge 1)_*} & [X, KU \wedge Q \wedge SG] & \xrightarrow{(1 \wedge j_Q \wedge 1)_*} & [X, \Sigma^3 KU \wedge SG] & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \kappa_0} & & \downarrow{\scriptstyle \kappa_0} & & & & \\
 & & \mathrm{Hom}(KU_0X, KU_0SG) & \xrightarrow{i_{Q**}} & \mathrm{Hom}(KU_0X, KU_0(Q \wedge SG)) & . & & &
\end{array}$$

Here the left vertical arrow $\kappa_0 = \kappa_0^{ku}$ and the right one $\kappa_1 = \kappa_1^{ku}$ are both isomorphisms because of (1.5) i), and the top and the bottom horizontal arrows j_{Q**} and i_{Q**} are also isomorphisms. Then a similar discussion to (1.20) shows that there exists a homomorphism

(1.21) $\rho: [X, \Sigma^3 KU \wedge SG] \to [X, KU \wedge Q \wedge SG]$

satisfying $(1 \wedge j_Q \wedge 1)_*\rho = 1$ and $(t_u \wedge 1)_*\rho = \rho(t_u \wedge 1)_*$.

As a dual of Lemma 1.1 we have

Lemma 1.2. *Let G be a divisible abelian group and $g': X \to \Sigma^1 KC \wedge Q \wedge SG$ be a map satisfying $(t_c \wedge 1)g' = g'$. If the composite $(\eta \wedge 1)(\tau\pi_c^{-1}$*

$\wedge 1)(1\wedge j_Q\wedge 1)g' : X \to \Sigma^6 KO\wedge SG$ is trivial, then there exist maps $h_1 : X \to \Sigma^4 KO \wedge SG$ and $g : X \to \Sigma^1 KC\wedge Q\wedge SG$ such that $(1\wedge i_Q \wedge 1)h_1 = (\tau\pi_C^{-1}\wedge 1)g$, $(t_C\wedge 1)g = g$ and $(\zeta\wedge 1)g = (\zeta\wedge 1)g'$.

Proof. Choose a map $h_0 : X \to \Sigma^4 KO\wedge SG$ satisfying $(\varepsilon_u\wedge 1)h_0 = (\zeta\wedge 1)(1\wedge j_Q\wedge 1)g'$, and hence a map $\ell' : X\to \Sigma^5 KU\wedge SG$ such that $(\varepsilon_C\wedge 1)h_0 = (1\wedge j_Q\wedge 1)g' + (\gamma\pi_u\wedge 1)\ell'$. Then it follows that $2(\gamma\pi_u\wedge 1)\ell' = 0$ because $t_C\varepsilon_C = \varepsilon_C$ and $t_C\gamma = -\gamma$. By applying the right inverse ρ of $(1\wedge j_Q\wedge 1)_*$ obtained in (1.21) we verify that the composite $(\gamma\pi_u\wedge 1)\rho(\ell') : X\to \Sigma^1 KC\wedge Q\wedge SG$ has order 2. Set $g = g' + (\gamma\pi_u\wedge 1)\rho(\ell')$, then its map satisfies the equalities $(\varepsilon_C\wedge 1)h_0 = (1\wedge j_Q\wedge 1)g$, $(t_C\wedge 1)g = g$ and $(\zeta\wedge 1)g = (\zeta\wedge 1)g'$. Obviously the first equality implies that there exists a map $h_1 : X\to \Sigma^4 KO\wedge SG$ with $(1\wedge i_Q\wedge 1)h_1 = (\tau\pi_C^{-1}\wedge 1)g$.

2. Pure projective and 2-torsion free.

2.1. In this section we will only deal with a CW-spectrum X such that KU_*X is pure projective and 2-torsion free, and hence KC_*X is expressed as in (1.9). We denote by A'_E the image of the induced homomorphism $(\varepsilon_C\tau\eta)_* : KC_{-1}X \to KC_1X$ where $KC_1X \cong (D\oplus F)\oplus(A\otimes Z/2\oplus B\oplus C)$. Then

(2.1) $A'_E = (\varepsilon_C\tau\eta)_*(KC_{-1}X) = (\varepsilon_C\tau)_*(E\otimes Z/2) \subset A\otimes Z/2 \subset KC_1X$

because $\eta_*(KC_{-1}X) = E\otimes Z/2$ and $\eta_*(KC_0X) = A\otimes Z/2$ by $(1.10)_i$ ($i = 3, 0$).

Since $\tau\varepsilon_C = \eta : \Sigma^1 KO\to KO$, it follows immediately that $\tau_*A'_E = 0 = (\tau\pi_C^{-1})_*A'_E$. Choose subgroups A'_D and A' in $A\otimes Z/2$ so that $A\otimes Z/2 \cong A'_E\oplus A'_D\oplus A'$ and $\mathrm{Ker}(\varepsilon_C\tau)_*|_{A\otimes Z/2} \cong A'_E\oplus A'$. Thus the subgroups A'_D and A' satisfy

(2.2) $(\varepsilon_C\tau)_* : A'_D \xrightarrow{\cong} (\varepsilon_C\tau)_*(A\otimes Z/2) = D'_A$ *and* $(\varepsilon_C\tau)_*A' = 0$.

We moreover put

(2.3) $A''_0 = (\varepsilon_C\eta)_*(KO_0X)$ *and* $A''_4 = (\pi_C\varepsilon_C\eta)_*(KO_{-4}X)$

both of which are subgroups of $A\otimes Z/2$. It is obvious that

(2.4) $A'_E \subset A''_0 \cap A''_4$ *and* $(\varepsilon_C\tau)_*A''_0 = 0 = (\varepsilon_C\tau)_*A''_4$

because $\varepsilon_C\tau\eta = \pi_C\varepsilon_C\eta\tau\pi_C^{-1} : \Sigma^2 KC\to KC$ and $\tau\pi_C\varepsilon_C = 0 : \Sigma^5 KO\to KO$.

More precisely we have

(2.5) i) $\tau_* A''_0 = \eta^2_*(KO_0X)$, $(\tau\pi_C^{-1})_* A''_4 = \eta^2_*(KO_{-4}X)$ *and*
ii) $(\tau\pi_C^{-1})_* A''_0 = 0 = \tau_* A''_4$.

Lemma 2.1. *There exists a direct sum decomposition*

$$A \otimes Z/2 \cong A'_E \oplus A'_D \oplus A'_0 \oplus A'_4$$

with $A'_E \oplus A'_0 \cong (\varepsilon_C \eta)_*(KO_0X)$ *and* $A'_E \oplus A'_4 \cong (\pi_C \varepsilon_C \eta)_*(KO_{-4}X)$.
Similarly there exist direct sum decompositions

$$D \otimes Z/2 \cong D'_A \oplus D'_B \oplus D'_1 \oplus D'_5,\ B \otimes Z/2 \cong B'_D \oplus B'_E \oplus B'_2 \oplus B'_6 \text{ and}$$
$$E \otimes Z/2 \cong E'_B \oplus E'_A \oplus E'_3 \oplus E'_7$$

with suitable isomorphisms as above.

Proof. We will prove only the $A \otimes Z/2$ case. Choose subgroups A'_0 and A'_4 in $A \otimes Z/2$ so that $A'_E \oplus A'_0 \cong A''_0$ and $A'_E \oplus A'_4 \cong A''_4$. It is sufficient to show that $\mathrm{Ker}(\varepsilon_C \tau)_*|_{A \otimes Z/2} \cong A'_E \oplus A'_0 \oplus A'_4$. First, take an element $x \in A \otimes Z/2$ with $(\varepsilon_C \tau)_* x = 0$. Using the equality $\eta^2 = \tau\varepsilon_C\eta$: $\Sigma^2 KO \to KO$, we get elements $u \in KO_0X$ and $v \in KO_5X$ such that $x = (\varepsilon_C\eta)_*u + (\pi_C^{-1}\varepsilon_C)_*v$. Moreover we notice that $\varepsilon_{u*}v = 0$ because $\zeta_*(A \otimes Z/2) = 0$. This implies that the element x is contained in $A''_0 + A''_4 = A'_E + A'_0 + A'_4$. Thus it is verified that $\mathrm{Ker}(\varepsilon_C\tau)_*|_{A \otimes Z/2} \cong A'_E + A'_0 + A'_4$.

Next we take elements $a \in A'_0$, $b \in A'_4$ and $c \in A'_E$ satisfying $a + b + c = 0$. Then it follows from (2.5) ii) that $\tau_* a = 0 = (\tau\pi_C^{-1})_* b$. Since $a \in (\varepsilon_C\eta)_*(KO_0X)$ and $b \in (\pi_C\varepsilon_C\eta)_*(KO_{-4}X)$ we use (2.5) i) to find elements x and y in $KC_{-1}X$ such that $a = (\varepsilon_C\eta\tau)_*x$ and $b = (\pi_C\varepsilon_C\eta\tau\pi_C^{-1})_*y = (\varepsilon_C\eta\tau)_*y$. Since the elements a and b are both belonging to A'_E, they must be zero, thus $a = b = c = 0$. Consequently it is shown that $\mathrm{Ker}(\varepsilon_C\tau)_*|_{A \otimes Z/2} \cong A'_E \oplus A'_0 \oplus A'_4$.

2.2. We now choose a direct sum decomposition

(2.6) $A \cong A_E \oplus A_D \oplus A_0 \oplus A_4$ *with* A_E, A_D *and* A_4 *free*,

which after tensored with $Z/2$ gives the direct sum decomposition $A \otimes Z/2 \cong A'_E \oplus A'_D \oplus A'_0 \oplus A'_4$ obtained in Lemma 2.1 (use [Ku]). Similarly we choose direct sum decompositions

(2.7) $D \cong D_A \oplus D_B \oplus D_1 \oplus D_5$, $B \cong B_D \oplus B_E \oplus B_2 \oplus B_6$ *and*

$$E \cong E_B \oplus E_A \oplus E_3 \oplus E_7,$$

which after tensored with $Z/2$ are respectively the direct sum decompositions of $D \otimes Z/2$, $B \otimes Z/2$ and $E \otimes Z/2$ obtained in Lemma 2.1.

Set $G = A_D \cong D_A$, which is free. We denote by $i_A : G \to A$ and $i_D : G \to D$ the canonical inclusions with $i_A(G) = A_D$ and $i_D(G) = D_A$. Let $t_0 : A \oplus C \to D \oplus E \otimes Z/2 \oplus F$ and $t_1 : D \oplus F \to A \otimes Z/2 \oplus B \oplus C$ be the homomorphisms given in (1.12), which are determined by the conjugations t_{c*} on KC_iX ($i = 0, 1$) respectively.

Lemma 2.2. *There exist unique homomorphisms* $r : G \to D \oplus F$ *and* $s : G \to B \oplus C$ *satisfying* $t_0 i_A - i_D = 2r$ *and* $t_1 i_D = 2s$.

Proof. First we use the canonical inclusion $i_A : G \to KC_0X \cong (A \oplus C) \oplus (D \oplus E \otimes Z/2 \oplus F)$. By use of $(1.10)_0$ we observe that the composite $\eta_* i_A : G \to KC_1X \cong (D \oplus F) \oplus (A \otimes Z/2 \oplus B \oplus C)$ is identified with the canonical projection $G = A_D \to A_D \otimes Z/2 \cong A'_D$. So the composite $(\eta \varepsilon_c \tau)_* i_A : G \to KC_2X \cong (B \oplus C) \oplus (D \otimes Z/2 \oplus E \oplus F)$ is factorized through D'_A because $(\varepsilon_c \tau)_* : A'_D \xrightarrow{\cong} D'_A$ by (2.2). Therefore the composite $(\varepsilon_c \tau)_* i_A : G \to KC_1X$ is written into the form of a sum $-i_D + 2u + v + w$ for some homomorphisms $u : G \to D$, $v : G \to F$ and $w : G \to A \otimes Z/2 \oplus B \oplus C$ (use $(1.10)_1$). Then it follows from $(1.11)_0$ that the composite $(\gamma \zeta \varepsilon_c \tau)_* i_A : G \to KC_0X$ coincides with the sum $-i_D + 2(u+v)$. Since $\gamma \zeta \varepsilon_c \tau = 1 - t_c : KC \to KC$, it is easily checked that $t_0 i_A = i_D + 2r$ setting $r = -(u+v) : G \to D \oplus F$.

Next, use the canonical inclusion $i_D : G \to KC_1X \cong (D \oplus F) \oplus (A \otimes Z/2 \oplus B \oplus C)$ in place of $i_A : G \to KC_0X$. The composite $(\eta \varepsilon_c \tau)_* i_D : G \to KC_3X \cong (E \oplus F) \oplus (A \oplus B \otimes Z/2 \oplus C)$ is trivial because $\tau_* D'_A = \tau_* \eta_* A'_D = 0$ by use of (2.2) and $(1.10)_1$. By a parallel discussion to the above we can find a homomorphism $s : G \to B \oplus C$ such that $(\gamma \zeta \varepsilon_c \tau)_* i_D = -2s : G \to KC_1X$. Use the equality $\gamma \zeta \varepsilon_c \tau = 1 - t_c$ again to obtain the desired one $t_1 i_D = 2s$.

Let $f_G : \Sigma^1 Q \wedge SG \to KU \wedge X$ be the map whose induced homomorphisms $\kappa_{ku}(f_G)_* : KU_{i-1}(Q \wedge SG) \to KU_iX$ ($i = 0, 1$) are given by the canonical inclusions $i_A : G \to A \oplus B \oplus C \oplus C$ and $i_D : G \to D \oplus E \oplus F \oplus F$ respectively. Since $(t_u \wedge 1) f_G = f_G$, we obtain a map $g_G : \Sigma^1 Q \wedge SG \to KC \wedge X$ with $(\zeta \wedge 1) g_G = f_G$ by use of the cofiber sequence (1.1) iii). According to (1.15) the induced homomorphisms $\kappa_0(g_G) : KC_{-1}(Q \wedge SG) \to KC_0X$ and

$\kappa_1(g_G): KC_0(Q\wedge SG) \to KC_1X$ are respectively given by the following matrices

$$(2.8)\quad \begin{aligned} &\begin{pmatrix} i_A & 0 \\ z & i_D \end{pmatrix} : G\oplus G \to (A\oplus C)\oplus(D\oplus E\otimes Z/2\oplus F) \\ &\begin{pmatrix} i_D & 0 \\ w & [i_A] \end{pmatrix} : G\oplus(G\otimes Z/2) \to (D\oplus F)\oplus(A\otimes Z/2\oplus B\oplus C) \end{aligned}$$

for some homomorphisms $z: G\to D\oplus E\otimes Z/2\oplus F$ and $w: G\to A\otimes Z/2 \oplus B\oplus C$. In particular, take $z=r$ and $w=s$, both of which are obtained in Lemma 2.2. Then (1.16) shows that the given map g_G satisfies $(t_C\wedge 1)g_G = g_G$. Thus we have

Corollary 2.3. *There exists a map $g_G: \Sigma^1 Q\wedge SG \to KC\wedge X$ such that $(\zeta\wedge 1)g_G = f_G$ and $(t_C\wedge 1)g_G = g_G$.*

2.3. We will now prove one of our main theorems.

Theorem 2.4. *Let X be a CW-spectrum such that KU_*X is pure projective and 2-torsion free, thus it is a direct sum of a free group and cyclic p-groups ($p\neq 2$). Then there exist abelian groups A_i ($0\leqq i\leqq 7$), C_j ($0\leqq j\leqq 1$) and G_k ($0\leqq k\leqq 3$) so that X is quasi KO_*-equivalent to the wedge sum $(\bigvee_i \Sigma^i SA_i)\vee(\bigvee_j \Sigma^j P\wedge SC_j)\vee(\bigvee_k \Sigma^{k+1}Q\wedge SG_k)$ where C_j and G_k are taken to be free* (cf. [B, Theorem 3.2]).

Proof. Using the abelian groups chosen in (2.6) and (2.7) we set $A_1 = D_1$, $A_2=B_2$, $A_3=E_3$, $A_5=D_5$, $A_6=B_6$, $A_7=E_7$, $C_0=C$ and $C_1=F$, and moreover $G_0=A_D\cong D_A$, $G_1=D_B\cong B_D$, $G_2=B_E\cong E_B$ and $G_3 = E_A\cong A_E$. Abbreviate by Y the required wedge sum $(\bigvee_i \Sigma^i SA_i)\vee(\bigvee_j \Sigma^j P \wedge SC_j)\vee(\bigvee_k \Sigma^{k+1}Q\wedge SG_k)$. It is obvious that $KU_*Y\cong KU_*X$ on both of which the conjugations t_{u*} behave as the same action. For each component Y_H of the wedge sum Y we choose a map $f_H: Y_H\to KU\wedge X$ whose induced homomorphism $\kappa_{KU}(f_H)_*: KU_*Y_H\to KU_*X$ is the canonical inclusion. Here H is taken to be A_i ($0\leqq i\leqq 7$), C_j ($0\leqq j\leqq 1$) and G_k ($0\leqq k\leqq 3$). Since $(t_u\wedge 1)f_H = f_H$ by virtue of [Y3, Lemma 1.2], there exists a map $g_H: Y_H\to KC\wedge X$ satisfying $(\zeta\wedge 1)g_H = f_H$ for each H. Along the line adopted in [Y2, Y3] we will find a map $h_H: Y_H\to KO\wedge X$ such that $(\varepsilon_u\wedge 1)h_H = f_H$, and then apply [Y2, Proposition 1.1] to show that the map $h=$

$\bigvee_H h_H : Y = \bigvee_H Y_H \to KO \wedge X$ is a quasi KO_*-equivalence.

We will only find such maps h_H in the cases $H = A_0$, C_0 and G_0. The other cases are similarly done.

i) The $H = A_0$ case: The induced homomorphism $(\eta\tau\pi_C^{-1})_* : KC_0X \to KO_6X$ restricted to A_0 is trivial since $(\tau\pi_C^{-1})_*A'_0 = 0$ by (2.5) ii). Hence the composite $(\eta\wedge 1)(\tau\pi_C^{-1}\wedge 1)g_{A_0} = (\varepsilon_O\pi_U^{-1}\wedge 1)f_{A_0} : SA_0 \to \Sigma^2 KO \wedge X$ becomes trivial because A_0 is written as a direct sum of a free group and a uniquely 2-divisible group. So we get a required map $h_{A_0} : SA_0 \to KO\wedge X$ with $(\varepsilon_u\wedge 1)h_{A_0} = f_{A_0}$.

ii) The $H = C_0$ case: Since $\eta\wedge 1 : \Sigma^1 KO \wedge P \to KO \wedge P$ is trivial, it is immediate that the composite $(\eta\wedge 1)(\tau\pi_C^{-1}\wedge 1)g_{C_0} = (\varepsilon_O\pi_U^{-1}\wedge 1)f_{C_0} : P\wedge SC_0 \to \Sigma^2 KO\wedge X$ becomes trivial. So we get a required map $h_{C_0} : P\wedge SC_0 \to KO\wedge X$ with $(\varepsilon_u\wedge 1)h_{C_0} = f_{C_0}$.

iii) The $H = G_0$ case: For simplicity we put $G = G_0$, $f = f_{G_0}$ and $g = g_{G_0}$ where $G = A_D \cong D_A$ and it is free. By Corollary 2.3 the map $g : \Sigma^1 Q\wedge SG \to KC\wedge X$ can be chosen to satisfy $(t_C\wedge 1)g = g$ as well as $(\zeta\wedge 1)g = f$. The induced homomorphism $(\eta\tau\pi_C^{-1})_* : KC_1X \to KO_{-1}X$ restricted to D_A is trivial since $(\tau\pi_C^{-1})_*D'_A = 0$. So the composite $(\eta\wedge 1)(\tau\pi_C^{-1}\wedge 1)g(i_Q\wedge 1) : SG \to \Sigma^1 KO\wedge X$ becomes trivial. By applying Lemma 1.1 we then get a map $h_1 : \Sigma^1 SG \to KO\wedge X$ such that $h_1(j_Q\wedge 1) = (\tau\pi_C^{-1}\wedge 1)g$ although the map g with $(t_C\wedge 1)g = g$ and $(\zeta\wedge 1)g = f$ might be changed slightly for the new one.

In order to observe that the composite $(\varepsilon_O\pi_U^{-1}\wedge 1)f = (\eta\wedge 1)(\tau\pi_C^{-1}\wedge 1)g = (\eta\wedge 1)h_1(j_Q\wedge 1) : Q\wedge SG \to \Sigma^1 KO\wedge X$ is trivial, we will next show that there exists a map $k : SG \to KO\wedge X$ satisfying $(\eta^2\wedge 1)k = (\eta\wedge 1)h_1$ as in the proof of [Y2, Theorem 3.4]. Denote by $i_A : G \to A$ and $i_D : G \to D$ the canonical inclusions with $i_A(G) = A_D$ and $i_D(G) = D_A$ respectively. Moreover we note that the conjugation $t_{C*} = \begin{pmatrix} 1 & 0 \\ t_0 & -1 \end{pmatrix}$ on $KC_0X \cong (A\oplus C)\oplus(D\oplus E\otimes Z/2\oplus F)$ for a certain homomorphism $t_0 : A\oplus C \to D\oplus E\otimes Z/2\oplus F$ (see (1.12)). By (2.8) we may express as $\chi_{KC}(g)_* = \chi_0(g) = \begin{pmatrix} i_A & 0 \\ z & i_D \end{pmatrix} : KC_3(Q\wedge SG) \cong G\oplus G \to KC_4X \cong (A\oplus C)\oplus(D\oplus E\otimes Z/2\oplus F)$. Here the homomorphism $z : G \to D\oplus E\otimes Z/2\oplus F$ satisfies $t_0 i_A = 2z + i_D$ by virtue of (1.16) because $(t_C\wedge 1)g = g$. Recall [Y2, (3.5)] that the induced homomorphism $\varepsilon_{C*} : KO_3(Q\wedge SG) \to KC_3(Q\wedge SG)$ is represented by the column $\begin{pmatrix} 2 \\ 1 \end{pmatrix} : G \to G\oplus G$ (cf. (1.13)). Then an easy computation shows that the composite $\chi_{KC}(g)_*\varepsilon_{C*}: KO_3(Q\wedge SG)\to KC_4X$ coincides with the composite $(1+t_{C*})i_A: G\to(A\oplus C)\oplus(D\oplus E\otimes Z/2\oplus F)$. Since $\tau\pi_C^{-1}t_C=-\tau\pi_C^{-1}$, it is easily checked that the composite $(\tau\pi_C^{-1})_*\chi_{KC}(g)_*\varepsilon_{C*}: KO_3(Q\wedge SG)\to KO_1X$ is trivial. Thus $\chi_{KO}((\tau\pi_C^{-1}\wedge 1)g)_*=\chi_{KO}(h_1(j_Q\wedge 1))_*: KO_3(Q\wedge SG)\to KO_1X$ is trivial.

Use the commutative diagram

$$\begin{array}{ccccc}
[SG,\Sigma^{-2}KC\wedge X] & \xleftarrow{(\eta\wedge 1)_*} & [SG,\Sigma^{-1}KC\wedge X] & \xrightarrow{(j_Q\wedge 1)^*} & [\Sigma^{-3}Q\wedge SG,\Sigma^{-1}KC\wedge X]\\
\downarrow\chi_0 & & \downarrow\chi_0 & & \downarrow\chi_0\\
\mathrm{Hom}(KO_0SG,KC_2X) & \xleftarrow[\eta_{**}]{} & \mathrm{Hom}(KO_0SG,KC_1X) & \xrightarrow[(j_{Q*})^*]{} & \mathrm{Hom}(KO_3(Q\wedge SG),KC_1X)
\end{array}$$

where the left two vertical arrows $\chi_0=\chi_0^{ko}$ are isomorphisms since G is free. Notice that $\chi_0((\varepsilon_C\wedge 1)h_1(j_Q\wedge 1)): KO_3(Q\wedge SG)\to KC_1X$ is trivial. So we see that $\chi_0((\varepsilon_C\wedge 1)h_1): KO_0SG\to KC_1X$ has order 2 since the bottom right horizontal arrow $(j_{Q*})^*$ is just multiplication by 2 on $\mathrm{Hom}(G,KC_1X)$. In other words, $\chi_0((\varepsilon_C\wedge 1)h_1): G\to D\oplus F\oplus A\otimes Z/2\oplus B\oplus C$ is factorized through $A\otimes Z/2$. Then $\chi_0((\eta\varepsilon_C\wedge 1)h_1): KO_0SG\to KC_2X$ becomes trivial because $\eta_*(A\otimes Z/2)=0$. Therefore the composite $(\eta\varepsilon_C\wedge 1)h_1:\Sigma^2SG\to KC\wedge X$ is trivial since the left vertical arrow χ_0 is an isomorphism. So we get a map $k: SG\to KO\wedge X$ satisfying $(\eta^2\wedge 1)k=(\eta\wedge 1)h_1$. Consequently there exists a map $h:\Sigma^1Q\wedge SG\to KO\wedge X$ with $(\varepsilon_u\wedge 1)h=f$ as desired.

3. Pure injective and 2-divisible.

3.1. In this section we will only deal with a CW-spectrum X such that KU_*X is pure injective and 2-divisible, and hence KC_*X is expressed as in (1.9). Denote by A'_E the image of the induced homomorphism $(\varepsilon_C\tau\eta)_*: KC_1X\to KC_3X$ where $KC_3X\cong(D*Z/2\oplus E\oplus F)\oplus(A\oplus C)$. Thus

(3.1) $A'_E=(\varepsilon_C\tau\eta)_*(KC_1X)=(\varepsilon_C\tau)_*(E*Z/2)\subset A*Z/2\subset KC_3X$

because $\eta_*(KC_1X)=E*Z/2$ and $\eta_*(KC_2X)=A*Z/2$ by $(1.10)_i$ $(i=1,2)$.

Since $\tau_*A'_E=0$, we can choose subgroups A'_D and A' in $A*Z/2$ so that $A*Z/2\cong A'_E\oplus A'_D\oplus A'$ and $\mathrm{Ker}(\varepsilon_C\tau)_*|_{A*Z/2}\cong A'_E\oplus A'$. Thus the subgroups A'_D and A' satisfy

(3.2) $(\varepsilon_C\tau)_*: A'_D\xrightarrow{\cong}(\varepsilon_C\tau)_*(A*Z/2)=D'_A$ *and* $(\varepsilon_C\tau)_*A'=0$.

As a dual of Lemma 2.1 we have

Lemma 3.1. *There exists a direct sum decomposition*

$$A * Z/2 \cong A'_E \oplus A'_D \oplus A'_0 \oplus A'_4$$

with $A'_E \oplus A'_0 \cong (\varepsilon_C \eta)_*(KO_2X)$ *and* $A'_E \oplus A'_4 \cong (\pi_C \varepsilon_C \eta)_*(KO_{-2}X)$.
Similarly there exist direct sum decompositions

$$D * Z/2 \cong D'_A \oplus D'_B \oplus D'_1 \oplus D'_5,\ B * Z/2 \cong B'_D \oplus B'_E \oplus B'_2 \oplus B'_6 \text{ and}$$
$$E * Z/2 \cong E'_B \oplus E'_A \oplus E'_3 \oplus E'_7$$

with suitable isomorphisms as above.

Proof. Choose subgroups A'_0 and A'_4 in $A * Z/2$ so that $A'_E \oplus A'_0 \cong (\varepsilon_C \eta)_*(KO_2X)$ and $A'_E \oplus A'_4 \cong (\pi_C \varepsilon_C \eta)_*(KO_{-2}X)$. Then we can easily show that $\mathrm{Ker}(\varepsilon_C \tau)_*|_{A*Z/2} \cong A'_E \oplus A'_0 \oplus A'_4$ by the quite same argument as in the proof of Lemma 2.1.

We now choose a direct sum decomposition

(3.3) $A \cong A_E \oplus A_D \oplus A_0 \oplus A_4$ *with* A_E, A_D *and* A_4 *divisible* 2-*torsion*,

which restricted to the torsion subgroups of order 2 is just the direct sum decomposition $A * Z/2 \cong A'_E \oplus A'_D \oplus A'_0 \oplus A'_4$ obtained in Lemma 3.1. Similarly we choose direct sum decompositions

(3.4) $D \cong D_A \oplus D_B \oplus D_1 \oplus D_5,\ B \cong B_D \oplus B_E \oplus B_2 \oplus B_6$ *and*
$E \cong E_B \oplus E_A \oplus E_3 \oplus E_7$

which induce respectively the direct sum decompositions of $D * Z/2$, $B * Z/2$ and $E * Z/2$ obtained in Lemma 3.1.

Set $G = A_D \cong D_A$, which is divisible 2-torsion. We denote by $p_A : A \to G$ and $p_D : D \to G$ the canonical projections with $p_A(A_D) = G$ and $p_D(D_A) = G$. Let $t_0 : A \oplus B * Z/2 \oplus C \to D \oplus F$ and $t_3 : D * Z/2 \oplus E \oplus F \to A \oplus C$ be the homomorphisms given in (1.12), which are determined by the conjugations t_{C*} on KC_iX $(i = 0, 3)$ respectively. By a dual argument to the proof of Lemma 2.2 we show

Lemma 3.2. *There exist unique homomorphisms* $r : A \oplus C \to G$ *and* $s : E \oplus F \to G$ *satisfying* $p_D t_0 - p_A = 2r$ *and* $p_A t_3 = 2s$.

Proof. First we use the canonical projection $p_D : KC_0X \cong (A \oplus B * Z/2 \oplus C) \oplus (D \oplus F) \to G$. By means of $(1.10)_i$ $(i = 2, 3)$ and (3.2) we observe as in the proof of Lemma 2.2 that the composite $p_D(\varepsilon_C\tau)_*: KC_{-1}X \cong (D*Z/2\oplus E\oplus F)\oplus(A\oplus C)\to G$ is written into the form of a sum $p_A+2x+y+z$ for some homomorphisms $x: A\to G$, $y: C\to G$ and $z: D*Z/2\oplus E\oplus F\to G$. Then $(1.11)_3$ shows that the composite $p_D(\varepsilon_C\tau\gamma\zeta)_*: KC_0X\to G$ is identified with the sum $p_A+2(x+y)$. Since $\varepsilon_C\tau\gamma\zeta=1+t_C: KC\to KC$, it is easily checked that $p_Dt_0=p_A+2r$ setting $r=x+y: A\oplus C\to G$.

Using the canonical projection $p_A: KC_3X\cong(D*Z/2\oplus E\oplus F)\oplus(A\oplus C)\to G$ in place of $p_D: KC_0X\to G$, we can similarly find a homomorphism $s: E\oplus F\to G$ such that $p_A(\varepsilon_C\tau\gamma\zeta)_*=2s: KC_3X\to G$. This equality implies the desired one $p_At_3=2s$ because $\varepsilon_C\tau\gamma\zeta=1+t_C$.

Let $f_G: X\to\Sigma^1KU\wedge Q\wedge SG$ be the map whose induced homomorphisms $\kappa_{KU}(f_G)_*: KU_iX\to KU_{i-1}(Q\wedge SG)$ $(i=0, 1)$ are given by the canonical projections $p_A: A\oplus B\oplus C\oplus C\to G$ and $p_D: D\oplus E\oplus F\oplus F\to G$ respectively. Since $(t_u\wedge 1)f_G=f_G$, there exists a map $g_G: X\to\Sigma^1KC\wedge Q\wedge SG$ with $(\zeta\wedge 1)g_G=f_G$. According to (1.18) the induced homomorphisms $\kappa_0(g_G): KC_0X\to KC_{-1}(Q\wedge SG)$ and $\kappa_3(g_G): KC_3X\to KC_2(Q\wedge SG)$ are respectively given by the following matrices

$$(3.5)\quad \begin{aligned}&\begin{pmatrix}p_A & 0\\ w & p_D\end{pmatrix}: (A\oplus B*Z/2\oplus C)\oplus(D\oplus F)\to G\oplus G\\ &\begin{pmatrix}p_D & 0\\ z & p_A\end{pmatrix}: (D*Z/2\oplus E\oplus F)\oplus(A\oplus C)\to(G*Z/2)\oplus G\end{aligned}$$

for some homomorphisms $w: A\oplus B*Z/2\oplus C\to G$ and $z: D*Z/2\oplus E\oplus F\to G$. In particular, take $w=-r$ and $z=-s$ by using the homomorphisms r and s obtained in Lemma 3.2. Then (1.19) asserts that the given map g_G satisfies $(t_C\wedge 1)g_G=g_G$. Thus we have

Corollary 3.3. *There exists a map $g_G: X\to\Sigma^1KC\wedge Q\wedge SG$ such that $(\zeta\wedge 1)g_G=f_G$ and $(t_C\wedge 1)g_G=g_G$.*

3.2. We will finally prove another main result, which is a dual of Theorem 2.4.

Theorem 3.4. *Let X be a CW-spectrum such that KU_*X is pure injective and 2-divisible, thus it is a direct summand of a direct product of a divisible group and cyclic p-groups $(p\neq 2)$. Then there exist abelian groups A_i $(0\leqq i\leqq 7)$, C_j $(0\leqq j\leqq 1)$ and G_k $(0\leqq k\leqq 3)$ so that X is* *quasi KO_*-equivalent to the wedge sum* $(\bigvee_i \Sigma^i SA_i) \vee (\bigvee_j \Sigma^j P \wedge SC_j) \vee (\bigvee_k \Sigma^{k+1} Q \wedge SG_k)$ *where C_j and G_k are taken to be divisible 2-torsion* (cf. [B, Theorem 3.3]).

Proof. As in the proof of Theorem 2.4 we take A_i, C_j and G_k to be the abelian groups chosen in (3.3) and (3.4). For each component Y_H of the required wedge sum $Y = (\bigvee_i \Sigma^i SA_i) \vee (\bigvee_j \Sigma^j P \wedge SC_j) \vee (\bigvee_k \Sigma^{k+1} Q \wedge SG_k)$, we choose a map $f_H : X \to KU \wedge Y_H$ whose induced homomorphism $\kappa_{ku}(f_H)_* : KU_*X \to KU_* Y_H$ is the canonical projection. Since $(t_u \wedge 1)f_H = f_H$, there exists a map $g_H : X \to KC \wedge Y_H$ satisfying $(\zeta \wedge 1)g_H = f_H$ for each H. By a dual argument to the proof of Theorem 2.4 we will only show that there exist maps $h_H : X \to KO \wedge Y_H$ such that $(\varepsilon_u \wedge 1)h_H = f_H$ in the cases $H = A_0$, C_0 and G_0.

i) The $H = A_0$ case: Use the commutative diagram

$$\begin{array}{ccccccccc}
0 & \to & \mathrm{Ext}(KO_6X, A_0) & \to & [\Sigma^{-3}X, KO \wedge SA_0] & \xrightarrow{\kappa_4} & \mathrm{Hom}(KO_7X, A_0) & \to & 0 \\
 & & \downarrow{\scriptstyle (\eta_*)^*} & & \downarrow{\scriptstyle (\eta \wedge 1)^*} & & \downarrow{\scriptstyle (\eta_*)^*} & & \\
0 & \to & \mathrm{Ext}(KO_5X, A_0) & \to & [\Sigma^{-2}X, KO \wedge SA_0] & \xrightarrow{\kappa_4} & \mathrm{Hom}(KO_6X, A_0) & \to & 0
\end{array}$$

involving the Anderson universal coefficient sequences (1.5) ii), where $\kappa_4 = \kappa_4^{ko}$. The induced homomorphism $\kappa_{ko}((\eta \wedge 1)g_{A_0})_* : KO_6X \to KC_7SA_0$ is trivial because $(\pi_C^{-1}\varepsilon_C\eta)_*(KO_6X) \cong A'_E \oplus A'_4 \subset A_E \oplus A_4$ by Lemma 3.1. This implies that $\kappa_4((\eta \wedge 1)(\tau\pi_C^{-1} \wedge 1)g_{A_0}) = 0$ in $\mathrm{Hom}(KO_6X, A_0)$. Obviously the composite $(\eta \wedge 1)(\tau\pi_C^{-1} \wedge 1)g_{A_0}$ has order 2. However $\mathrm{Ext}(KO_5X, A_0)$ is uniquely 2-divisible since A_0 is a direct sum of a divisible group and a uniquely 2-divisible group. So we see that the composite $(\eta \wedge 1)(\tau\pi_C^{-1} \wedge 1)g_{A_0} = (\varepsilon_O\pi_u^{-1} \wedge 1)f_{A_0} : X \to \Sigma^2 KO \wedge SA_0$ is in fact trivial. Hence there exists a required map $h_{A_0} : X \to KO \wedge SA_0$ with $(\varepsilon_u \wedge 1)h_{A_0} = f_{A_0}$.

ii) The $H = C_0$ case: The composite $(\eta \wedge 1)(\tau\pi_C^{-1} \wedge 1)g_{C_0} = (\varepsilon_O\pi_u^{-1} \wedge 1)f_{A_0} : X \to \Sigma^2 KO \wedge P \wedge SC_0$ is evidently trivial. So we get a required map $h_{A_0} : X \to KO \wedge P \wedge SC_0$ with $(\varepsilon_u \wedge 1)h_{C_0} = f_{C_0}$.

iii) The $H = G_0$ case: For simplicity we put $G = G_0$, $f = f_{G_0}$, $g = g_{G0}$ where $G = A_D \cong D_A$ and it is divisible 2-torsion. By virtue of Corollary 3.3 the map $g: X \to \Sigma^1 KC \wedge Q \wedge SG$ can be chosen to satisfy $(t_C \wedge 1)g = g$ as well as $(\zeta \wedge 1)g = f$. Denote by $p_A : A \to G$ and $p_D : D \to G$ the canonical projections with $p_A(A_D) = G$ and $p_D(D_A) = G$. According to

(3.5), $\chi_{KC}(g)_* = \chi_0(g) = \begin{pmatrix} p_A & 0 \\ w & p_D \end{pmatrix} : KC_0X \to KC_{-1}(Q \wedge SG)$ and $\chi_{KC}(g)_* = \chi_3(g) = \begin{pmatrix} p_D & 0 \\ z & p_A \end{pmatrix} : KC_3X \to KC_2(Q \wedge SG)$ for some homomorphisms $w : A \oplus B * Z/2 \oplus C \to G$ and $z : D * Z/2 \oplus E \oplus F \to G$. As is easily checked, the induced homomorphism $\chi_{KO}((\eta \wedge 1)g)_* = \chi_{KC}(g)_*(\varepsilon_C \eta)_* : KO_2X \to KC_2(Q \wedge SG)$ is trivial because $(\varepsilon_C \eta)_*(KO_2X) \cong A'_E \oplus A'_0 \subset A_E \oplus A_0$ and $p_A(A_E \oplus A_0 \oplus A_4) = 0$. Hence the composite $(\eta \wedge 1)(\tau\pi_C^{-1} \wedge 1)(1 \wedge j_Q \wedge 1)g : X \to \Sigma^6 KO \wedge SG$ becomes trivial since $\chi_{-4}^{ko} : [\Sigma^{-6}X, KO \wedge SG] \to \mathrm{Hom}(KO_2X, G)$ is an isomorphism by (1.5) ii). By applying Lemma 1.2 we then get a map $h_1 : X \to \Sigma^4 KO \wedge SG$ such that $(1 \wedge i_Q \wedge 1)h_1 = (\tau\pi_C^{-1} \wedge 1)g$ although the map g with $(t_C \wedge 1)g = g$ and $(\zeta \wedge 1)g = f$ might be changed slightly for the new one.

As in the latter part of the proof iii) of Theorem 2.4 we will next show that there exists a map $k : X \to \Sigma^5 KO \wedge SG$ satisfying $(\eta^2 \wedge 1)k = (\eta \wedge 1)h_1$, in order to observe that the composite $(\varepsilon_O \pi_U^{-1} \wedge 1)f = (\eta \wedge 1)\cdot(\tau\pi_C^{-1} \wedge 1)g = (1 \wedge i_Q \wedge 1)(\eta \wedge 1)h_1 : X \to \Sigma^3 KO \wedge Q \wedge SG$ is trivial. By (1.12) we note that the conjugation $t_{C*} = \begin{pmatrix} 1 & 0 \\ t_0 & -1 \end{pmatrix}$ on $KC_0X \cong (A \oplus B * Z/2 \oplus C) \oplus (D \oplus F)$ for a certain homomorphism $t_0 : A \oplus B * Z/2 \oplus C \to D \oplus F$. Then (1.19) says that the homomorphism $w : A \oplus B * Z/2 \oplus C \to G$ satisfies $p_D t_0 = p_A - 2w$ where $\chi_{KC}(g)_* = \chi_0(g) = \begin{pmatrix} p_A & 0 \\ w & p_D \end{pmatrix}$, because $(t_C \wedge 1)g = g$. Recall that $(\tau\pi_C^{-1})_* : KC_{-1}(Q \wedge SG) \to KO_{-4}(Q \wedge SG)$ is represented by the row $(-1 \;\; 2) : G \oplus G \to G$ (cf. [Y2, (3.5)]). Then an easy computation shows that the composite $(\tau\pi_C^{-1})_* \chi_{KC}(g)_* : KC_0X \to KO_{-4}(Q \wedge SG)$ coincides with the composite $p_D(1 - t_{C*}) : (A \oplus B * Z/2 \oplus C) \oplus (D \oplus F) \to G$. So the composite $(\tau\pi_C^{-1})_* \chi_{KC}(g)_* \varepsilon_{C*} : KO_0X \to KO_{-4}(Q \wedge SG)$ becomes trivial because $t_C \varepsilon_C = \varepsilon_C$. Thus $\chi_{KO}((\tau\pi_C^{-1} \wedge 1)g)_* = \chi_{KO}((1 \wedge i_Q \wedge 1)h_1)_* : KO_0X \to KO_{-4}(Q \wedge SG)$ is trivial.

Since the induced homomorphism $i_{Q*} : KO_{-4}SG \to KO_{-4}(Q \wedge SG)$ is multiplication by 2 on G, it follows immediately that $\chi_{KO}(h_1)_* : KO_0X \to KO_{-4}SG$ has order 2. Hence the composite $\chi_{KO}(h_1)_*(\tau\pi_C^{-1})_* : KC_3X \cong (D * Z/2 \oplus E \oplus F) \oplus (A \oplus C) \to KO_{-4}SG \cong G$ is factorized through $D * Z/2$. Since $\eta_*(KC_2X) = A * Z/2$, the composite $\chi_{KO}((\eta \wedge 1)h_1)_*(\tau\pi_C^{-1})_* : KC_2X \to KO_{-4}SG$ is trivial, too. We here use the commutative diagram

$$\begin{array}{ccccc}
[\Sigma^{-1}X, \Sigma^4 KO\wedge SG] & \xrightarrow{(\eta^2\wedge 1)^*} & [\Sigma^1 X, \Sigma^4 KO\wedge SG] & & \\
\downarrow{\scriptstyle \varkappa_0} & & \downarrow{\scriptstyle \varkappa_0} & & \\
\mathrm{Hom}(KO_1X, KO_{-4}SG) & \xrightarrow[(\eta_*^2)^*]{} & \mathrm{Hom}(KO_{-1}X, KO_{-4}SG) & \xrightarrow[(\tau\pi_{c*}^{-1})^*]{} & \mathrm{Hom}(KC_2X, KO_{-4}SG)
\end{array}$$

where the two vertical arrows $\varkappa_0 = \varkappa_0^{ko}$ are isomorphisms by (1.5) ii). As is easily seen, there exists a map $k: X \to \Sigma^5 KO\wedge SG$ satisfying $(\eta^2\wedge 1)k = (\eta\wedge 1)h_1$. Consequently we obtain a map $h: X \to \Sigma^1 KO\wedge Q\wedge SG$ with $(\varepsilon_u\wedge 1)h = f$ as desired.

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