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REMARKS ON HOMOTOPY GROUPS OF SYMMETRIC SPACES

Dedicated to Professor Shôrô Araki on his 60th birthday

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1. **Introduction.** We denote by $O_n(\mathbb{F})$ the classical group O_n, U_n or Sp_n if $\mathbb{F}=\mathbb{R}$(real), $\mathbb{C}$(complex) or $\mathbb{H}$(quaternionic), respectively. Let $SO_n\subset O_n$ be the rotation group and $\Gamma_n=SO_{2n}/U_n$. The homotopy groups $\pi_{2n+i}(\Gamma_n)$ for $i\leq 5$ were determined except for the following cases ([2], [5], [12]) : $i=2, n\equiv 0 \bmod 4$; $i=5, n\equiv 0,1,3 \bmod 4$. The purpose of the present note is to try determining the group structures in these cases. We denote by (a, b) the greatest common divisor of integers a and b. Our result with Kachi's one is stated as follows.

Theorem 1. i) $\pi_{8n+2}(\Gamma_{4n})\approx \mathbb{Z}_{8(3,2n+1)}$.
ii) $\pi_{8n+6}(\Gamma_{4n+2})\approx \mathbb{Z}_{2(3,2n+2)}$.

Theorem 2. i) $\pi_{8n+5}(\Gamma_{4n})\approx \mathbb{Z}_{(4n+2)!(6,n)/12}\oplus \mathbb{Z}_2$.
ii) $\pi_{8n+11}(\Gamma_{4n+3})\approx \mathbb{Z}$.

Our method is first to use the homotopy exact sequence of a triad $(SO_{2n}; SO_{2n-2k}, U_n)$ [3]. To determine a group extension, we shall use Mimura's lemma about Toda brackets in fibrations.

2. **Proof of Theorem 1.** We denote the Stiefel manifold $O_n(\mathbb{F})/O_{n-k}(\mathbb{F})$ by $V_{n,k}$ or $W_{n,k}$ according as $\mathbb{F}=\mathbb{R}$ or $\mathbb{C}$. $\mathbb{R}P^n_k=\mathbb{R}P^n/\mathbb{R}P^{k-1}$ stands for the stunted space of the real n-dimensional projective space $\mathbb{R}P^n$. Let $\eta_n\in\pi_{n+1}(S^n)$ for $n\geq 2$ and $\nu_n\in\pi_{n+3}(S^n)$ for $n\geq 4$ be the Hopf maps and $\eta^2_n=\eta_n\circ\eta_{n+1}\in\pi_{n+2}(S^n)$.

By [5], the odd component of $\pi_{4n+2}(\Gamma_{2n})$ is isomorphic to $\mathbb{Z}_{(3,n+1)}$. So we shall work in the 2-components. By [2], $\pi_{8n+i}(\Gamma_{4n+3})\approx\pi_{8n+i+1}(SO)$ for $i\leq 4$. Therefore, by use of the homotopy exact sequence of a triad $(SO_{8n+5}; SO_{4n}, U_{4n+2})$ [3], we have $\pi_{8n+2}(\Gamma_{4n})\approx\pi_{8n+3}(\Gamma_{4n+3},\Gamma_{4n})=\pi_{8n+3}(V_{8n+5,5}, W_{4n+2,2})$. $(V_{8n+5,5}, \mathbb{R}P^{8n+4}_{8n})$ and $(W_{4n+2,2}, S^{8n+1}\vee S^{8n+3})$ are highly connected and $\mathbb{R}P^{8n+4}_{8n}=\Sigma^{8n}\mathbb{R}P^4_0$. So we have

$$\pi_{8n+3}(V_{8n+5,5}, W_{4n+2,2})=\pi_{8n+3}(\Sigma^{8n}\mathbb{R}P^4_0, S^{8n+1}\vee S^{8n+3})$$

$$
\begin{aligned}
&= \pi_{8n+3}(S^{8n} \vee (S^{8n+2} \cup_{\lambda} e^{8n+4})) \\
&\approx \pi_{8n+3}(S^{8n}) \approx \mathbf{Z}_8,
\end{aligned}
$$

where $\lambda = \eta_{8n+2}$. This completes the proof of i) of Theorem 1.

A proof of ii) was given by Kachi [5]. We note the following:

$$
\begin{aligned}
\pi_{8n+6}(\Gamma_{4n+2}) \approx \pi_{8n+7}(\Gamma_{4n+5}, \Gamma_{4n+2}) &= \pi_{8n+7}(V_{8n+9,5}, W_{4n+4,2}) \\
&= \pi_{8n+7}(\Sigma^{8n}\mathbf{R}P_4^8, S^{8n+5} \vee S^{8n+7}) \\
&= \pi_{8n+7}((S^{8n+4} \vee S^{8n+6}) \cup_{\mu} e^{8n+8}) \approx \mathbf{Z}_2,
\end{aligned}
$$

where $\mu = \nu_{8n+4} \vee \eta_{8n+6}$.

By use of our method, we can prove the following

Theorem 3 (Ōshima [12]). $\pi_{2n+1}(\Gamma_n) \approx \mathbf{Z}_{n!/2} \oplus \mathbf{Z}_2$ *if* $n \equiv 2 \bmod 4$.

Proof. It suffices to give a proof for $n \geq 6$. We consider the following natural map between exact sequences :

$$
\begin{array}{ccccccccc}
 & & & & & & \mathbf{Z}_{n!/2} & & \\
 & & & & & & \wr\wr & & \\
0 & \longrightarrow & \pi_{2n+2}(S^{2n}) & \xrightarrow{\partial} & \pi_{2n+1}(\Gamma_n) & \longrightarrow & \pi_{2n+1}(\Gamma_{n+1}) & \longrightarrow & 0 \\
 & & \| & & \downarrow j_* & & \downarrow & & \\
 & & \pi_{2n+2}(\Gamma_{n+1}, \Gamma_n) & \xrightarrow{\partial'} & \pi_{2n+1}(\Gamma_n, \Gamma_{n-1}) & \rightarrow & \pi_{2n+1}(\Gamma_{n+1}, \Gamma_{n-1}) & & \\
 & & \wr\wr & & \wr\wr & & & & \\
 & & \mathbf{Z}_2 & & \mathbf{Z}_{24} & & & &
\end{array}
$$

$\pi_{2n+1}(\Gamma_{n+1}, \Gamma_{n-1}) = \pi_{2n+1}(V_{2n+1,3}, W_{n,1}) = \pi_{2n+1}(\mathbf{R}P_{2n-2}^{2n}, S^{2n-1}) = \pi_{2n+1}(S^{2n-2} \cup_{\lambda} e^{2n}) \approx \mathbf{Z}_{12}$, where $\lambda = \eta_{2n-2}$. This isomorphism is obtained from the relation $\eta_n^3 = 12\nu_n$. So ∂' is nontrivial. Assume that the upper sequence does not split and let α be a generator of $\pi_{2n+1}(\Gamma_n)$. Then, $\partial(\eta_{2n}^2) = (n!/2)\alpha$, and so $\partial'(\eta_{2n}^2) = j_*\partial(\eta_{2n}^2) = (n!/2)j_*(\alpha) = 0$. This is a contradiction and completes the proof.

Remark. $\Omega(V_{2n+2t-1,2t-1}, W_{n+t-1,t-1})$ is identified with a fiber $\Gamma_{n,t}$ of the inclusion $\Gamma_n \hookrightarrow \Gamma_{n+t}$ [5].

3. A relation among the characteristic maps. In this section we shall state an application of Theorem 1. Let $\gamma_n'(\mathbf{F}) \in \pi_{d(n+1)-2}(O_n(\mathbf{F}))$ be the characteristic map, where $d = \dim_{\mathbf{R}} \mathbf{F}$. Let $r: U_n \hookrightarrow SO_{2n}$ and $c: Sp_n \hookrightarrow U_{2n}$ be the canonical inclusions and let x_n be an integer such that $(x_n, 6) = (n, 2)(n+1, 3)$. We set $\alpha_n = (3+(-1)^{n+1})/2\ rc\gamma_n'(\mathbf{H}) + x_n\gamma_{4n}'(\mathbf{R}) \circ \nu_{4n-1}$

for $n \geq 2$.

Theorem 4. $r(\gamma'_{2n}(\mathbf{C}) \circ \eta^2_{4n}) = (3+(-1)^n)\alpha_n$ *for* $n \geq 2$. *In particular,* $r(\gamma'_{4n}(\mathbf{C}) \circ \eta^2_{8n}) = 12/(3, 2n+1)\ rc\gamma'_{2n}(\mathbf{H})$.

Proof. By Lemma 1.6 of [6] and its proof and by Corollary 24.5 of [14], we have $\pi_{4n+2}(U_{2n}) = \mathbf{Z}_{(2n+1)!}\{c\gamma'_n(\mathbf{H})\} \oplus \mathbf{Z}_2\{\gamma'_{2n}(\mathbf{C}) \circ \eta^2_{4n}\}$ for $n \geq 2$. By [6], $\pi_{8n+2}(SO_{8n}) \approx \mathbf{Z}_{24} \oplus \mathbf{Z}_8$. By [1] and [13], $\pi_{8n+6}(SO_{8n+4}) = \pi_{8n+6}(SO) \oplus \pi_{8n+7}(V_{8n+9,5}) \approx \mathbf{Z}_{48} \oplus \mathbf{Z}_4$ for $n \geq 2$. By [4], $\pi_{14}(SO_{12}) \approx \mathbf{Z}_{24} \oplus \mathbf{Z}_4$.

We consider the following commutative diagram for $n \geq 2$:

$$
\begin{array}{ccccc}
 & & \pi_{4n+2}(U_{2n}) & \overset{r'_*}{\searrow} & \pi_{4n+3}(S^{4n+1}) \\
 & & \downarrow r_* & & \downarrow \partial' \\
\pi_{4n+3}(S^{4n}) & \xrightarrow{\partial} & \pi_{4n+2}(SO_{4n}) & \xrightarrow{i_*} & \pi_{4n+2}(SO_{4n+1}) \\
 & \overset{\partial''}{\searrow} & \downarrow p_* & & \downarrow i'_* \\
 & & \pi_{4n+2}(\Gamma_{2n}) & & \pi_{4n+2}(SO_{4n+2}),
\end{array}
$$

where the mappings are canonical and the horizontal and perpendicular sequences are exact. By [2], $\pi_{4n+2}(\Gamma_{2n+1}) = 0$ for $n \geq 2$. Therefore r'_* and ∂'' are epimorphisms, and so are i_* and p_*. $\pi_{4n+2}(SO_{4n+k}) \approx \mathbf{Z}_{4(3-k)}$ for $k=1$ or 2 [6]. Since $\partial'\eta^2_{4n+1} = \gamma'_{4n+1}(\mathbf{R}) \circ \eta^2_{4n} = r'\gamma'_{2n}(\mathbf{C}) \circ \eta^2_{4n}$, $r'c\gamma'_n(\mathbf{H})$ is taken as a generator of $\pi_{4n+2}(SO_{4n+1})$ and we have a relation $4r'c\gamma'_n(\mathbf{H}) = r'\gamma'_{2n}(\mathbf{C}) \circ \eta^2_{4n}$. So $rc\gamma'_n(\mathbf{H})$ and $\partial\nu_{4n} = \gamma'_{4n}(\mathbf{R}) \circ \nu_{4n-1}$ generate $\pi_{4n+2}(SO_{4n})$ and $r\gamma'_{2n}(\mathbf{C}) \circ \eta^2_{4n} \equiv 4rc\gamma'_n(\mathbf{H}) \bmod \gamma'_{4n}(\mathbf{R}) \circ \nu_{4n-1}$. Let x'_n be an integer such that $r\gamma'_{2n}(\mathbf{C}) \circ \eta^2_{4n} = 4rc\gamma'_n(\mathbf{H}) + x'_n\gamma'_{4n}(\mathbf{R}) \circ \nu_{4n-1}$ and $1 \leq x'_n \leq 24$. Then x'_n is a multiple of 4 or 2 according as $n \equiv 0$ or $1 \bmod 2$. We set $x_n = x'_n/(3+(-1)^n)$. By Theorem 1, we have the assertion of the theorem. This completes the proof.

4. Proof of Theorem 2. We shall prove ii) of Theorem 2. We consider exact sequences :

$$
\begin{array}{ccccccc}
\pi_{8n+12}(S^{8n+6}) & \xrightarrow{\Delta} & \pi_{8n+11}(\Gamma_{4n+3}) & \rightarrow & \pi_{8n+11}(\Gamma_{4n+4}) & \rightarrow & 0\ ; \\
\wr\wr & & & & \wr\wr & & \\
\mathbf{Z}_2\{\nu^2_{8n+6}\} & & & & \mathbf{Z} & &
\end{array}
$$

$$
\begin{array}{ccccccc}
\pi_{8n+9}(S^{8n+6}) & \xrightarrow{\Delta} & \pi_{8n+8}(\Gamma_{4n+3}) & \rightarrow & \pi_{8n+8}(\Gamma_{4n+4}) & \rightarrow & \pi_{8n+8}(S^{8n+6}). \\
 & & \wr\wr & & \wr\wr & & \wr\wr \\
 & & \mathbf{Z}_{2(12,2n+1)} & & \mathbf{Z}_2 \oplus \mathbf{Z}_2 & & \mathbf{Z}_2
\end{array}
$$

The order of $\Delta\nu_{8n+6}$ must be $(12, 2n+1)$. So, $\Delta(\nu_{8n+6}^2) = \Delta\nu_{8n+6}\circ\nu_{8n+8} = 0$. This completes the proof of ii).

We shall prove i) of Theorem 2. By Propositions 12.1 and 12.2 of [5] and by [15], we have the assertion for $n=1$. We shall use the following group structures for $n \geq 2$:

$\pi_{8n+4}(U_{4n}) \approx \mathbf{Z}_{(4n+2)!(6,n)/12}$ [8] ;
$\pi_{8n+8}(U_{4n+1}) \approx \mathbf{Z}_{(4n+4)!(12,2n+1)/24} \oplus \mathbf{Z}_2$ [9] ;
$\pi_{8n+8}(U_{4n}) \approx \mathbf{Z}_{(4n+4)!/c} \oplus (\mathbf{Z}_2)^3$ [11] ;
$\pi_{8n+4}(SO_{8n}) = 0$ [6] ;
$\pi_{8n+5}(SO_{8n}) \approx (\mathbf{Z}_2)^2$, $\quad \pi_{8n+8}(SO_{8n+2}) \approx \mathbf{Z}_{240} \oplus (\mathbf{Z}_2)^2$ and
$\pi_{8n+8}(SO_{8n}) \approx (\mathbf{Z}_2)^8$ ([1], [7]),

where c is the complex James number $W\{4n+5, 5\}$ and $(\mathbf{Z}_2)^k = \mathbf{Z}_2 \oplus \cdots \oplus \mathbf{Z}_2$ (k times).

We consider the natural map between exact sequences:

$$\begin{array}{ccccccc}
(\mathbf{Z}_2)^2 & & & & & & \mathbf{Z}_{240} \\
\wr\wr & & & & & & \wr\wr \\
\pi_{8n+9}(S^{8n+1}) & \xrightarrow{\Delta} & \pi_{8n+8}(U_{4n}) & \longrightarrow & \pi_{8n+8}(U_{4n+1}) & \longrightarrow & \pi_{8n+8}(S^{8n+1}) \\
\downarrow & & \downarrow r_* & & \downarrow & & \downarrow \\
\pi_{8n+9}(V_{8n+2,2}) & \xrightarrow{\Delta'} & \pi_{8n+8}(SO_{8n}) & \rightarrow & \pi_{8n+8}(SO_{8n+2}) & \rightarrow & \pi_{8n+8}(V_{8n+2,2}). \\
\wr\wr & & & & & & \wr\wr \\
(\mathbf{Z}_2)^5 & & & & & & \mathbf{Z}_{240} \oplus (\mathbf{Z}_2)^2
\end{array}$$

We remark that $\pi_{8n+9}(V_{8n+2,2}) = \pi_{8n+9}(S^{8n}) \oplus \pi_{8n+9}(S^{8n+1}) \approx \mathbf{Z}_2\{\nu_n^3\} \oplus (\mathbf{Z}_2)^4$ [15] and Δ and Δ' are split monomorphisms. So, from this diagram, we have the following.

$(*)$ $\qquad \gamma'_{8n}(\mathbf{R}) \circ \nu_{8n-1}^3 \neq 0$ *and it is not in the image of* r_*.

We consider an exact sequence

$$\pi_{8n+6}(S^{8n}) \xrightarrow{\partial} \pi_{8n+5}(\Gamma_{4n}) \rightarrow \pi_{8n+5}(\Gamma_{4n+1}) \rightarrow 0.$$
$$\pi_{8n+5}(\Gamma_{4n+1}) \approx \mathbf{Z}_{(4n+2)!(6,n)/12}$$

$\partial\nu_{8n}^2 = p\circ\gamma'_{8n}(\mathbf{R})\circ\nu_{8n-1}^2$, where $p: SO_{8n} \rightarrow \Gamma_{4n}$ is the projection. By $(*)$, it is nontrivial. So we have $\pi_{8n+5}(\Gamma_{4n}) \approx \mathbf{Z}_{(4n+2)!(6,n)/12} \oplus \mathbf{Z}_2$ or $\mathbf{Z}_{(4n+2)!(6,n)/6}$. To settle the group extension, we need the following

Lemma 5 (Mimura [10]). *Let* $F \xrightarrow{i} X \xrightarrow{p} B$ *be a fibration. Suppose that*

$\alpha \in \pi_{m+1}(B)$, $\beta \in \pi_j(S^m)$ and $\gamma \in \pi_k(S^j)$ satisty the conditions $(\Delta\alpha)\circ\beta = \beta\circ\gamma = 0$. For any element δ of a Toda bracket $\{\Delta\alpha, \beta, \gamma\} \subset \pi_{k+1}(F)$, there exists an element $\varepsilon \in \pi_{j+1}(X)$ such that $p_*\varepsilon = \alpha\circ\sum\beta$ and $i_*\delta = \varepsilon\circ\sum\gamma$.

Now we consider an exact sequence

$$\begin{array}{ccccccc} \pi_{8n+5}(SO_{8n}) & \xrightarrow{p*} & \pi_{8n+5}(\Gamma_{4n}) & \xrightarrow{\Delta} & \pi_{8n+4}(U_{4n}) & \xrightarrow{r*} & \pi_{8n+4}(SO_{8n}). \\ \wr\wr & & & & \wr\wr & & \| \\ (\mathbf{Z}_2)^2 & & & & \mathbf{Z}_{(4n+2)!(6,n)/12} & & 0 \end{array}$$

Let $\alpha \in \pi_{8n+5}(\Gamma_{4n})$ be an element such that $\Delta\alpha$ is a generator of $\pi_{8n+4}(U_{4n})$. Let ι_n be the identity map of S^n and $a = (4n+2)!(6,n)/12$. By Lemma 5, for any element $\delta \in \{\Delta\alpha, a\iota_{8n+4}, \nu_{8n+4}\}$, there exists an element $\varepsilon \in \pi_{8n+5}(SO_{8n})$ such that $r_*\delta = \varepsilon\circ\nu_{8n+5}$ and $p_*\varepsilon = a\alpha$. Suppose that $p_*\varepsilon \neq 0$. Then $p_*\varepsilon = p_*(\gamma'_{8n}(\mathbf{R})\circ\nu^2_{8n-1})$. So there exists an element $\theta \in \pi_{8n+5}(U_{4n})$ such that $\varepsilon = \gamma'_{8n}(\mathbf{R})\circ\nu^2_{8n-1} + r_*\theta$. Therefore $0 = p_*(\varepsilon\circ\nu_{8n+5}) = p_*(\gamma'_{8n}(\mathbf{R})\circ\nu^3_{8n-1})$. This contradicts the assertion $(*)$. Hence we have $a\alpha = 0$. This completes the proof of i) of Theorem 2.

Finally we shall prove the following

Proposition 6. *$\pi_{8n+7}(\Gamma_{4n+1}) \approx \mathbf{Z} \oplus \mathbf{Z}_2$ for $n = 1$ and it is isomorphic to $\mathbf{Z} \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$ or $\mathbf{Z} \oplus \mathbf{Z}_4$ for $n \geq 2$.*

Proof. We consider an exact sequence :

$$\pi_{8n+8}(S^{8n+2}) \xrightarrow{\Delta} \pi_{8n+7}(\Gamma_{4n+1}) \to \pi_{8n+7}(\Gamma_{4n+2}) \to 0.$$

By [5], $\pi_{8n+7}(\Gamma_{4n+2}) \approx \mathbf{Z}$ for $n = 1$; $\approx \mathbf{Z} \oplus \mathbf{Z}_2$ for $n \geq 2$. So the non-triviality of Δ concludes the assertion.

Consider an exact sequence :

$$\pi_{8n+9}(SO_{8n+3}) \longrightarrow \pi_{8n+9}(\Gamma_{4n+2}) \longrightarrow \pi_{8n+8}(U_{4n+1}) \longrightarrow \pi_{8n+8}(SO_{8n+3}) \xrightarrow{p*} \pi_{8n+8}(\Gamma_{4n+2}).$$

By [1], [7] and [9], it becomes the following :

$$\mathbf{Z}_2 \oplus \mathbf{Z}_2 \to \mathbf{Z}_{(4n+4)!(12,2n+1)/12} \to \mathbf{Z}_{(4n+4)!(12,2n+1)/24} \oplus \mathbf{Z}_2 \to \mathbf{Z}_2 \oplus \mathbf{Z}_2 \to \mathbf{Z}_{2(12,2n+1)}.$$

Therefore p_* is an epimorphism on the 2-component. Consider a commutative diagram

$$\begin{array}{ccc} \pi_{8n+8}(SO_{8n+3}) & \xrightarrow{p*} & \pi_{8n+8}(\Gamma_{4n+2}) \\ & {\scriptstyle p'*}\searrow \quad \swarrow{\scriptstyle q*} & \\ & \pi_{8n+8}(S^{8n+2}) & \end{array}$$

By [1] and [7], $\pi_{8n+7}(SO_{8n+k}) \approx \boldsymbol{Z} \oplus (\boldsymbol{Z}_2)^{4-k}$ for $k = 2$ or 3. So p'_* is trivial. Therefore q_* is trivial. This shows the nontriviality of Δ and completes the proof.

REFERENCES

[1] M. G. BARRATT and M. E. MAHOWALD : The metastable homotopy of $O(n)$, Bull. Amer. Math. Soc. **70** (1964), 758−760.

[2] B. HARRIS : Some calculations of homotopy groups of symmetric spaces, Trans. Amer. Math. Soc. **106** (1963), 174−184.

[3] I. M. JAMES : Products between homotopy groups, Compositio Math. **23** (1971), 329−345.

[4] H. KACHI : On the homotopy groups of rotation groups R_n, J. Fac. Sci. Shinshu Univ. **3** (1968), 13−33.

[5] H. KACHI : Homotopy groups of symmetric spaces Γ_n, J. Fac. Sci. Shinshu Univ. **13** (1978), 103−120.

[6] M. A. KERVAIRE : Some nonstable homotopy groups of Lie groups, Illinois J. Math. **4** (1960), 161−169.

[7] M. E. MAHOWALD : The Metastable Homotopy of S^n, Mem. Amer. Math. Soc. **72** (1968).

[8] H. MATSUNAGA : The homotopy groups $\pi_{2n+i}(U(n))$ for $i = 3, 4$ and 5, Mem. Fac. Sci. Kyushu Univ. **A15** (1961), 72−81.

[9] H. MATSUNAGA : Applications of functional cohomology operations to the calculus of $\pi_{2n+i}(U(n))$ for $i = 6$ and 7, $n \geqq 4$, Mem. Fac. Sci. Kyushu Univ. **A17** (1963), 29−62.

[10] M. MIMURA : Quelques groupes d'homotopie mêtastables des espaces symêtriques $Sp(n)$ et $U(2n)/Sp(n)$, C. R. Acad. Sc. Paris **262** (1966), 20−21.

[11] H. ŌSHIMA : Some James numbers of Stiefel manifolds, Math. Proc. Camb. Phil. Soc. **92** (1982), 139−161.

[12] H. ŌSHIMA : A homotopy group of the symmetric space $SO(2n)/U(n)$, Osaka J. Math. **21** (1984), 473−475.

[13] G. F. PAECHTER : The groups $\pi_r(V_{n,m})$ (I), Quart. J. Math. Oxford (2) **7** (1956), 249−268.

[14] N. E. STEENROD : The Topology of the Fibre Bundles, Princeton Univ. 1951.

[15] H. TODA : Composition Methods in Homotopy Groups of Spheres, Ann. of Math. Studies **49**, Princeton, 1962.

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