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ON THE HIGHER HOMOTOPY ASSOCIATIVITY OF P-REGULAR HOPF SPACES

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1. Introduction. Throughout the paper we consider the product space of odd dimensional spheres

$$X = S^{2n_1-1}\times\cdots\times S^{2n_k-1}$$

which is localized at a prime p. It is well known that X is p-equivalent to an H-space for an odd prime p. At the prime 2, Adams [1] has shown that X is 2-equivalent to an H-space if and only if $n_i = 1, 2$ or 4 for each i. At the prime 3, Hemmi [4] has shown that X is 3-equivalent to a homotopy associative H-space if and only if X is 3-equivalent to a loop space $(S^1)^a\times(S^3)^b\times SU(3)^c$, where a, b and c are non-negative integers. In general, it is known that X is p-equivalent to an A_{p-1}-space for an odd prime p by [8]. Furthermore Hubbuck-Mimura [6] and Iwase [7] have shown that for any odd prime p, if X is p-equivalent to a 1-connected A_p-space, then $n_i\in\{2, \cdots, p\}$ for each i. Hence it is reasonable to conjecture that for any odd prime p, X is p-equivalent to an A_p-space if and only if X is p-equivalent to a loop space. In this paper we consider the case $p=5$. By passing to the universal cover of X which inherits the higher homotopy associativity assumption, it will become clear that all circles in X can be omitted without essential loss of generality and so the additional hypothesis is made that X is 1-connected.

The main result of this paper is stated as follows:

Theorem. *The product* $X = S^{2n_1-1}\times\cdots\times S^{2n_k-1}$ $(n_1<\cdots<n_k)$ *is* 5-*equivalent to a* 1-*connected* A_5-*space if and only if it is* 5-*equivalent to one of the following spaces:*

$$S^3,\ S^7,\ SU(3),\ Sp(2),\ SU(4),\ SU(5).$$

Corollary. *The product* X *is* 5-*equivalent to a* 1-*connected* A_5-*space if and only if it is* 5-*equivalent to a loop space.*

This paper is organized as follows. In §2 we prepare three lemmas which will be needed to prove the main theorem. §3 gives a proof of the main theorem.

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2. Preliminaries. Suppose that $X = S^{2n_1-1}\times\cdots\times S^{2n_k-1}(n_1<\cdots<n_k)$ is p-equivalent to a 1-connected A_p-space for an odd prime p. Then we call the set of integers $(n_1, \cdots, n_k)$ the *half type* of X. Clearly the wedge sum $S^{2n_1-1}\vee\cdots\vee S^{2n_k-1}$ gives a generating subspace of X in the sense of [7]. Hence by [7], there exists a space $Q(p)$ such that $XP(p-1)\subset Q(p)\subset XP(p)$, where $XP(n)$ is the n-th projective space of X. So we can obtain the following quotient algebra M of $K^*(Q(p)\,;\,Z_{(p)}) = K^*(Q(p))\otimes Z_{(p)}$ which is a polynomial algebra truncated at height $p+1$:

$$M \cong Z_{(p)}[u_1, \cdots, u_k]^{(p+1)},$$

where $Z_{(p)}$ is the set of integers localized at p. Furthermore M can be chosen to be closed under the action of the Adams operations ψ^k for all k. Since the integral homology of $Q(p)$ has no p-torsion, the Atiyah-Hirzebruch spectral sequence with $Z_{(p)}$-coefficient of [3] collapses. And hence we can obtain the following quotient algebra N of $H^*(Q(p)\,;\,Z_{(p)})$ which is the associated graded ring of M and a polynomial algebra truncated at hight $p+1$:

$$N \cong Z_{(p)}[y_1, \cdots, y_k]^{(p+1)}, \quad \deg y_i = 2n_i.$$

Then by Hubbuck [5], N admits the following Hubbuck operations :

$$S^q : N_n \to N_{n+q(p-1)} \qquad (q \geqq 0)$$
$$R^q(k) : N_n \to N_{n+q(p-1)} \qquad (q \geqq 0,\ k\in Z).$$

And these operations satisfy the following properties :

(1) $S^n(x) \equiv x^p \bmod p$ *for any* $x\in N_n$.

(2) *the Cartan formulae*

$$S^q(xy) = \sum_{i+j=q} S^i(x)S^j(y)$$
$$R^q(k)(xy) = \sum_{i+j=q} R^i(k)(x)R^j(k)(y) \quad \textit{for any } x, y\in N.$$

(3) *the generalized Adem relations*

Let k be prime to p. Then for $q\geqq 1$,

$$(1-k^{q(p-1)})S^q \equiv \sum_{i=1}^{q-1} k^{(q-i)(p-1)}p^iR^i(k)S^{q-i} \bmod p^q.$$

In particular, this implies that $S^1S^{q-1}\equiv qS^q \bmod p$. To prove the theorem by using the Hubbuck operations, we prepare the following lemmas. In the case $X = S^{2n-1}$, we consider the properties (1) and (3) of the Hubbuck operations. For dimensional reasons we obtain the following :

Lemma 1. *If S^{2n-1} is p-equivalent to an A_p-space, then n divides $p-1$.*

From now on, when we write y_n for a generator, it is assumed that this generator has dimension n, that is, twice the dimension is the degree. Suppose that rank $X \geqq 2$. Then we see that $n_k \leqq p$ by Hubbuck-Mimura [6] and Iwase [7]. Now we consider the case $n_k < p$.

Lemma 2. *If N has a generator of dimension q not dividing $p-1$, then N has a generator of dimension r and the following condition is satisfied:*

$$r+p-1 \equiv 0 \mod q.$$

Proof. The properties (1) and (3) imply that

$$S^1 S^{q-1}(y_q) \equiv q y_q^p \mod p.$$

Since S^1 satisfies the Cartan formulae, the element $S^{q-1}(y_q)$ contains an element of the form $y_q^a y_r$, where y_r is in the indecomposable module QN_r. It follows that there exists an element y_r in QN_r such that

$$S^{q-1}(y_q) \equiv \alpha y_q^a y_r + I.E. \mod p$$
$$y_q^a S^1(y_r) \equiv \beta y_q^p + I.E. \mod p,$$

where $I.E.$ stands for "independent elements" of the base of N and $\alpha, \beta \not\equiv 0 \mod p$ and $0 < a < p$. Therefore it follows that

$$S^1(y_r) \equiv \gamma y_q^{p-a} + I.E. \mod p,$$

where $\gamma \not\equiv 0 \mod p$. Hence we obtain that $r+p-1 \equiv 0 \mod q$ for dimensional reasons. This completes the proof of the lemma. **Q. E. D.**

Next we consider the case $n_k = p$.

Lemma 3. *If N has a generator y_p of dimension p, then N has generators y_i of dimension i for all i, $2 \leqq i \leqq p-1$.*

Proof. We apply the generalized Adem relations. Let k be $p-1$. Then we have that

$$\sum_{h=1}^{p-1} k^{(p-h)(p-1)} p^h R^h(k) S^{p-h}(y_p) \equiv (1-k^{p(p-1)}) S^p(y_p) \mod p^p.$$

Since $\nu_p(k^{p(p-1)}-1) = 2$ by [2], we deduce that

$$\sum_{h=1}^{2} k^{(p-h)(p-1)} p^h R^h(k) S^{p-h}(y_p) \equiv \lambda p^2 y_p^p \mod p^3,$$

where $\lambda \not\equiv 0 \mod p$. Considering the possibility of contribution to the element y_p^p by the mapping $R^1(k)$, we see that $S^{p-1}(y_p)$ contains an element of the form $y_p^{p-1}y_1$ by the property (2), where y_1 is in QN_1. However by our hypothesis of connectivity, $Q(p)$ is 2-connected and $QN_1 = 0$. Therefore we conclude that the above argument does not hold. Hence it follows that

$$R^2(k)S^{p-2}(y_p) \equiv \lambda' y_p^p$$

where $\lambda' \not\equiv 0 \mod p$. Using a similar argument to the above, we can deduce that the only possibility of contribution to the element y_p^p by the mapping $R^2(k)$ lies on the element $y_p^{p-2}y_2$ for dimensional reasons, where y_2 is in QN_2. It follows that there exists an element y_2 in QN_2 such that

$$S^{p-2}(y_2) \equiv \alpha_2 y_p^{p-2} y_2 + I.E. \mod p$$
$$y_p^{p-2} R^2(k)(y_2) \equiv \beta_2 y_p^p$$

where α_2, $\beta_2 \not\equiv 0 \mod p$. Then we have

$$R^2(k)(y_2) \equiv \gamma_2 y_p^2 + I.E. \mod p,$$

where $\gamma_2 \not\equiv 0 \mod p$. Since $S^1 S^{p-3} \equiv (p-2)S^{p-2} \mod p$ by the property (3), using the above again, we see that the only possibility of contribution to the element $y_p^{p-2}y_2$ by the mapping S^1 lies on the element of the form $y_p^{p-3}y_3$ for dimensional reasons, where y_3 is in QN_3. It follows that there exists an element y_3 in QN_3 such that

$$S^{p-3}(y_p) \equiv \alpha_3 y_p^{p-3} y_3 + I.E. \quad \mod p$$
$$y_p^{p-3} S^1(y_3) \equiv \beta_3 y_p^{p-2} y_2 + I.E. \mod p,$$

where α_3, $\beta_3 \not\equiv 0 \mod p$. Therefore we have

$$S^1(y_3) \equiv \gamma_3 y_p y_2 + I.E. \mod p,$$

where $\gamma_3 \not\equiv 0 \mod p$. Hence repeating the above argument, we deduce by induction that there exist elements y_i in QN_i for $3 \leqq i \leqq p-1$ such that

$$S^1(y_i) \equiv \gamma_i y_p y_{i-1} + I.E. \mod p,$$

where $\gamma_i \not\equiv 0 \mod p$. This completes the proof of the lemma. **Q. E. D.**

In the next section, we prove the main theorem by applying these three lemmas.

3. The proof of the theorem. Firstly we consider the sufficient condition of the theorem. In the case $X = S^{2n-1}$, Sullivan [9] has shown that for an odd prime p, X is p-equivalent to a loop space if and only if n divides $p-1$. Hence we see that S^3 and S^7 are 5-equivalent to loop spaces. It is well known that $SU(n)$ and $Sp(n)$ are loop spaces. Hence S^3, S^7, $SU(3)$, $SU(4)$ and $SU(5)$ are 5-equivalent to loop spaces, and hence A_5-spaces.

Secondly we consider the necessary condition of the theorem. When X is 5-equivalent to a 1-connected A_5-space, we recall from Hubbuck-Mimura [6] and Iwase [7] that rank $X \leqq 4$. By applying the three lemmas we deduce that only the following half types can occur:

$$2, 4, (2, 3), (2, 4), (2, 3, 4), (2, 3, 4, 5).$$

In fact, in the case rank $X = 1$, we deduce that $n_1 = 2$ or 4 by Lemma 1. In the case rank $X = 2$, we deduce that $(n_1, n_2) = (2, 3)$ or $(2, 4)$ by Lemmas 2 and 3. In the case rank $X = 3$, we deduce that $(n_1, n_2, n_3) = (2, 3, 4)$ by Lemmas 2 and 3. In the case rank $X = 4$, we deduce that $(n_1, n_2, n_3, n_4) = (2, 3, 4, 5)$ by Lemma 3. Summing up we see that X is one of the following:

$$S^3,\ S^7,\ S^3 \times S^5,\ S^3 \times S^7,\ S^3 \times S^5 \times S^7,\ S^3 \times S^5 \times S^7 \times S^9,$$

where it is known that $SU(n) \underset{(5)}{\simeq} S^3 \times \cdots \times S^{2n-1}$ for $2 \leqq n \leqq 5$ and $Sp(n) \underset{(5)}{\simeq} S^3 \times \cdots \times S^{4n-1}$ for $n = 1$ and 2. Consequently, this completes the proof of the theorem.

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