Math. J. Okayama Univ. 32 (1990), 147-151

A THEOREM ON RADICALS IN FUNCTOR CATEGORIES WITH APPLICATION TO TORSION THEORIES

YAO MUSHENG

In this paper, we obtain a result concerning radicals in functor categories and give an application to torsion theories.

1. Let $\mathbf{C}$ be a small preadditive category, and $\mathbf{Ab}$ the category of abelian groups. $(\mathbf{C}^0, \mathbf{Ab})$ denotes the category of all additive contravariant functors from $\mathbf{C}$ to $\mathbf{Ab}$. Kernels, cokernels and direct sums in $(\mathbf{C}^0, \mathbf{Ab})$ can be defined componentwise. Then $(\mathbf{C}^0, \mathbf{Ab})$ is a Grothendieck category. For details, we to refer [5].

The Jacobson radical $J(T)$ of an object T in a Grothendieck category is defined to be the subobject which is the intersection of all its maximal subobjects. If T has no maximal subobjects, then define $J(T) = T$. A subobject $S \leq T$ is said to be superfluous if for any subobject $N \leq T$, $S+N = T$ always implies $N = T$. We use the notation $S \ll T$ to denote that S is a superfluous subobject of T.

Lemma 1. *Let S, T be objects in a Grothendieck category. Then for any $f \in Mor(S, T)$, $f(J(S)) \leq J(T)$, where $Mor(S, T)$ is the set of morphisms from S to T.*

This is well known. The following two results are also known.

Lemma 2. *J commutes with finite direct sums.*

Lemma 3. *Let T be an object in a Grothendieck category. Then $\sum T_\alpha \leq J(T)$, where $T_\alpha \ll T$.*

Let $\mathbf{F} = (\mathbf{C}^0, \mathbf{Ab})$ be the functor category, and T an object of $\mathbf{F}$. If B is an object of $\mathbf{C}$, then $T(B)$ is an abelian group. Let a be an element of $T(B)$. We define a contravariant functor S from $\mathbf{C}$ to $\mathbf{Ab}$ as follows: $S(C) = \{T(f)(a) \mid f \in \mathrm{Mor}(C, B)\}$ for any $C \in \mathbf{C}$. If α is a morphism from C to C', then define $S(\alpha): S(C') \to S(C)$ to be the restriction of $T(\alpha)$. It is easy to see that S is an object of $\mathbf{F}$ and actually a subobject of T. We call S the subobject of T generated by the element a. It is easy to verify that every object T is the sum of its subobject which can be generated by one element.

Lemma 4. *Let* $\mathbf{F} = (\mathbf{C}^0, \mathbf{Ab})$, *and* T *a nonzero object of* $\mathbf{F}$. *If* B *is an object of* $\mathbf{C}$ *such that* $T(B) \neq 0$, *then for any nonzero element* $a \in T(B)$, *there is a maximal subobject* W *of* T *with respect to* $a \overline{\in} W(B)$.

Proof. Since $a \neq 0$, the set of subobjects $\{W_\alpha\}$ of T with the property $a \overline{\in} W_\alpha(B)$ is not empty (at least contains the zero subobject). Consider any chain of the set $\{W_\alpha\}$:

$$W_1 \leq W_2 \leq W_3 \leq \cdots \qquad (*)$$

It is clear that $\bigcup_{i=1}^{\infty} W_i$ is a subobject of T which does not contain a (i.e. $a \overline{\in} (\bigcup_{i=1}^{\infty} W_i)(B)$). So it is the upper bound of the chain $(*)$. By Zorn's lemma, there is a maximal element in $\{W_\alpha\}$.

Theorem 5. *Let* $\mathbf{F} = (\mathbf{C}^0, \mathbf{Ab})$. *Then* $J(T) = \sum \{T_\alpha | T_\alpha \ll T\}$ *for any object* T *in* $\mathbf{F}$.

Proof. By lemma 3, we need only to verify that $J(T) \leq \sum T_\alpha$. Let $a \in J(T)(B)$ for some $B \in \mathbf{C}$, and S the subobject of $J(T)$ generated by a. We want to show that S is superfluous. Let N be a subobject of T such that $S+N = T$. We may assume $S \nleq N$ which means $a \overline{\in} N(B)$. Then, there is a nonempty set $\{W_\alpha\}$ of subobjects of T such that $N \leq W_\alpha$ and $a \overline{\in} W_\alpha(B)$. By lemma 4, there is a maximal element W in the set $\{W_\alpha\}$. Obviously $S+W = T$. But now W is a maximal subobject of T. In fact, let K be a subobjeet of T such that $W \leq K$, then $S \leq K$ which implies $K = T$. According to the definition of radical, $S \leq W$, which is a contradiction. Therefore S is superfluous. Consequently, we obtain $J(T) \leq \sum T_\alpha$.

Projective objects in functor categories have similar properties to projective modules in module categories. The following two results can be found in [4]:

Proposition 6. *Let* P *be a projective object in* $\mathbf{F}$ *such that* $J(P) \ll P$. *Then*

$$End\, P/J(End\, P) \simeq End\, P/J(P)$$

where $End\, P$ *denotes the endomorphism ring of* P.

Proposition 7. *Let* P *be a projective object in* $\mathbf{F}$, *and* $E = End\, P$. *If*

f is an element of E, then $f \in J(E)$ if and only if Im $f \ll P$, where $J(E)$ is the Jacobson radical of E.

The next result is a corollary of Theorem 5:

Proposition 8. *Let $P \neq 0$ be a projective object in* **F**. *Then $J(P) \neq P$, or equivalently, P has at least one maximal subobject.*

Proof. Suppose $J(P) = P$. By Theorem 5, $P = \sum_I \{T_\alpha | T_\alpha \ll P\}$. There is a canonical epimorphism f: $\bigoplus_I T_\alpha \to P$. Since P is projective, there is g: $P \to \bigoplus T_\alpha$ such that $fg = 1_P$. The canonical projection π_α : $\bigoplus T_\alpha \to T_\alpha$ induces a morphism $g_\alpha = \pi_\alpha g$: $P \to T_\alpha$. Let τ_α be the injection of $T_\alpha \to P$, then $\tau_\alpha g_\alpha \in \text{End}\, P$. Since $T_\alpha \ll P$, Im $\tau_\alpha g_\alpha \ll P$. This implies that Im $\sum_{I'} \tau_\alpha g_\alpha \ll P$ for any finite subset $I' \subset I$. By Proposition 7, $\sum_{I'} \tau_\alpha g_\alpha \in J(\text{End}\, P)$. Hence $1_P - \sum_{I'} \tau_\alpha g_\alpha$ is an automorphism of P. Since $P \neq 0$, there is an object $B \in$ **C** such that $P(B) \neq 0$. Let a be a nonzero element of $P(B)$. $P(B) = \sum_I T_\alpha(B)$. It is easy to see $1_{P_B} = \sum_I (\tau_\alpha g_\alpha)_B$ and $g_{\alpha_B}(a) \neq 0$ for only finite numbers of α. Hence, there is a finite subset I' of I such that $(1_{P_B} - \sum_{I'} (\tau_\alpha g_\alpha)_B)(a) = 0$. This means $1_P - \sum_{I'} \tau_\alpha g_\alpha$ is not an automorphism which contradicts the fact mentioned above.

2. Let R be a ring with identity, and Mod-R the category of all unital right R-modules. Let σ be a hereditary torsion theory on Mod-R. The quotient category of Mod-R with respect to σ is denoted by Mod-R/σ. For a right R-module M, the σ-Jacobson radical (or simply σ-radical) of M is defined by

$$J_\sigma(M) = \cap \{N | M/N \text{ is a } \sigma\text{-cocritical } R\text{-module}\}.$$

If there is no such N, we define $J_\sigma(M) = M$. $J_\sigma(M)$ is a σ-pure submodule of M, i.e. $M/J_\sigma(M)$ is σ-torsionfree. It is easy to see that in the quotient category Mod-R/σ, $J_\sigma(M)$ coincides with $J(M)$, where $J(M)$ is the Jacobson radical of the object M. We say a submodule N of M is σ-superfluous if and only if there is no proper σ-pure submodule K of M such that $N^c + K$ is σ-dense in M, where N^c denotes the σ-closure of N in M. We use the notation $N \ll_\sigma M$ to denote that N is a σ-superfluous submodule of M. In the theory of modules, there is a well known result:

$$J(M) = \sum \{M_\alpha | M_\alpha \ll M\}.$$

One may ask if there is a relative version of this result. In general the sum of σ-superfluous submodules of M need not be a σ-pure submodule. However, we may ask that, with what condition, the σ-radical of M coincides with the closure of the sum of its σ-superfluous submodules, i.e. $J_\sigma(M) = (\sum M_\alpha)^c$, where $M_\alpha \ll_\sigma M$.

Lemma 9. *Let N be a submodule of M. Then N is a σ-superfluous submodule of M if and only if N is a superfluous subobject of M in the quotient category Mod-R/σ.*

Proof. Let $N \ll_\sigma M$ in Mod-R. Then $N_\sigma = (N^c)_\sigma$, where N_σ and $(N^c)_\sigma$ denote the σ-localization of N and N^c respectively. From the definition of σ-superfluous submodule, $N^c \ll_\sigma M$. Since the lattice of pure submodules of M is isomorphic to the lattice of subobjects of M_σ, N_σ is superfluous in M_σ.

Conversely, let $N_\sigma \ll M_\sigma$ in the category Mod-R/σ, and K a σ-pure submodule of M such that N^c+K is dense in M. Then $(N^c+K)_\sigma = M_\sigma$ which implies $N_\sigma+K_\sigma = M_\sigma$. So that $K_\sigma = M_\sigma$ which means that K is dense in M. But K is σ-pure, so $K = M$.

The following lemma is a result due to Freyd (see [2]).

Lemma 10. *Let $\mathbf{F}$ be a Grothendieck category. If $\mathbf{F}$ has a family of finitely generated projective generators, then $\mathbf{F}$ is equivalent to a functor category $(\mathbf{C}^0, \mathbf{Ab})$, where $\mathbf{C}$ is a suitable small preadditive category.*

Now we prove the following theorem :

Theorem 11. *Let σ be a hereditary torsion theory on Mod-R. If the quotient category Mod-R/σ has a family of finitely generated projective generators, then for any right R-module M, $J_\sigma(M) = (\sum M_\alpha)^c$, where M_α runs over all the σ-superfluous submodules of M.*

Proof. Let F be an equivalence functor from a Grothendieck category $\mathbf{C}$ to another Grothendieck category $\mathbf{D}$. For any object C in $\mathbf{C}$, the lattice of subobjects of C is isomorphic to the lattice of $F(C)$. Therefore, a subobject B of C is maximal iff $F(B)$ is a maximal subobject of $F(C)$. This means $J(F(C)) = F(J(C))$. It is also clear that a subobject B of C is superfluous iff $F(B)$ is a superfluous subobject of $F(C)$. By Lemma 10, the category Mod-R/σ is equivalent to a functor category. Therefore $J_\sigma(M) = (\sum M_\alpha)^c$ by using Theorem 5 and Lemma 9.

Corollary 12. *Let σ be a right perfect hereditary torsion theory on*

Mod-R. Then, for any right R-module M, $J_\sigma(M) = \sum \{M_\alpha | M_\alpha \ll_\sigma M\}$.

Proof. Since σ is right perfect, the quotient category Mod-R/σ has a finitely generated projective generator. By Theorem 11, $J_\sigma(M) = (\sum M_\alpha)^c$. We need only to show that $\sum M_\alpha$ is σ-pure in M. Let $x \in M$ such that $xI \subset \sum M_\alpha$, where I is a dense right ideal of R. Since σ is perfect, we may assume that I is finitely generated. So, there is a finite subset A such that $xI \subset \sum_A M_\alpha$. However, every finite sum of σ-superfluous submodules is still superfluous. Hence xI is σ-superfluous in M. The closure of xI is also σ-superfluous. It means that $x \in \sum M_\alpha$. Therefore $\sum M_\alpha$ is σ-pure in M.

Acknowledgements. The author is grateful to the Japan Society for Promotion of Science. This paper was finished while he was a JSPS fellow and visited Okayama University. The author also would like to express his hearty thanks to Professor H. Tominaga for his kind help and hospitality.

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DEPARTMENT OF MATHEMATICS
FUDAN UNIVERSITY
SHANGHAI, CHINA

(*Received September 8, 1989*)