Math. J. Okayama Univ. 32 (1990), 139-145

ON THE GENERALIZED t-FULL MODULES II

Dedicated to Professor Yoshiki Kurata on his 60th birthday

SHOJI MORIMOTO and KAZUO SHIGENAGA

Let R be a ring with identity and R-*mod* the category of unitary left R-modules. For a left exact radical t of R-*mod*, generalizing the notion of Goldman's t-supporting module [2], Shigenaga [8] has defined the notion of generalized t-full modules and studied its basic properties. In this paper, we shall treat generalized t-full modules for a left exact preradical t and point out that these modules are closely related to t-weakly divisible modules. Finally, we determine those left exact preradicals t for which every module is generalized t-full (Theorem 2.5).

Throughout this note R means a ring with identity and modules mean unitary left R-modules unless otherwise stated. We denote the category of modules by R-*mod* and the injective hull of $M \in R$-*mod* by $E(M)$. As for terminologies and basic properties concerning torsion theories and preradicals, we refer to [1], [3] and [9].

For each preradical r, we denote the r-*torsion* (resp. r-*torsionfree*) class by $T(r)$ (resp. $F(r)$). Now for two preradicals r and s, we shall say that r *is larger than* s denoted by $r \geq s$ if $r(M) \supseteq s(M)$ for all modules M.

In what follows, t means a left exact preradical for R-*mod* and $\mathscr{L}(t)$ means the left linear topology corresponding to t unless otherwise specified.

1. Generalized t-full modules.

Definition 1.1. Let r be a preradical for R-*mod*. We shall call a module M *generalized* r-*full* (in short, $g.r$-*full*) if $r(M) = r(N)$ and M/N is in $T(r)$ for every essential submodule N of M. Also we call a module A r-*cocritical* (resp. r-*full*) (cf. [1]) if A is r-torsionfree and $r(A/B) = A/B$ for every nonzero submodule (resp. essential submodule) B of A.

A module A is r-full if and only if A is in $F(r)$ and is $g.r$-full. If $t \geq Z$ (the singular torsion functor), then A is a $g.t$-full module if and only if $t(A) = t(B)$ for every essential submodule B of A.

Lemma 1.2. *Let M be a module and N its submodule. If $t(M)=t(N)$ and M/N is in $T(t)$, then N is essential in M.*

Proof. Let X be a submodule of M such that $N\oplus X$ is essential in M. Then $t(N\oplus X)=t(N)\oplus t(X)\subseteq t(M)=t(N)$ and hence $t(X)=O\ (:=\{0\})$. Since $X\simeq(N\oplus X)/N\subseteq M/N\in T(t)$, X is in $T(t)$. Thus $X=O$, which means that N is essential in M.

From this lemma, we have

Corollary 1.3. *For a t-torsionfree module M, the following conditions are equivalent:*

1) *M is uniform and g.t-full.*
2) *M is uniform and t-full.*
3) *M is t-cocritical.*

Proposition 1.4. *If a module M is g.t-full, then so is every submodule of M.*

Proof. (cf. [8, Proposition 2])

Proposition 1.5. *Let $\{M_\alpha\}_{\alpha\in\Lambda}$ be a family of g.t-full modules. Then $\sum_{\alpha\in\Lambda}\oplus M_\alpha$ is g.t-full.*

Proof. Let $M=\sum_{\alpha\in\Lambda}\oplus M_\alpha$ and let N be an essential submodule of M. For each $\alpha\in\Lambda$, since $N\cap M_\alpha$ is essential in M_α, $t(N\cap M_\alpha)=t(M_\alpha)$ and so $t(M)\cap M_\alpha\cap N=t(M_\alpha)$. Thus, $t(M_\alpha)\subseteq N$ and hence we have $t(M)=t(N)$. On the other hand, $\sum_{\alpha\in\Lambda}\oplus(M_\alpha/(N\cap M_\alpha))\xrightarrow{\pi}\left(\sum_{\alpha\in\Lambda}\oplus M_\alpha\right)/N=M/N\to O$ is exact, where $\pi((m_\alpha+N\cap M_\alpha))=\sum m_\alpha+N$. Hence, M/N is in $T(t)$ and M is g.t-full.

Proposition 1.6. *Let M be a module. Then $E(M)$ is g.t-full if and only if $t(E(N))=t(N)$ and $E(N)/N$ is in $T(t)$ for every submodule N of $E(M)$.*

Proof. (cf. [8, Proposition 5])

Definition 1.7. Let r be a preradical for R-*mod*. We shall call a module H *weakly* r-*divisible* (resp. r-*divisible*) if the functor $Hom_R(-, H)$ preserves the exactness for every exact sequence $O \to A \to B \to C \to O$ of modules with $B \in T(r)$ (resp. $C \in T(r)$).

Proposition 1.8. [4, Lemma 2.2.] *Let r be a preradical and let M be a module and N its submodule. Then*

(1) *If M is weakly r-divisible and $r(M) \subseteq N$, then N is also weakly r-divisible.*

(2) *If r is idempotent, N is essential in M and is weakly r-divisible, then $r(M) \subseteq N$.*

In this paper, weakly divisible (resp. divisible) modules mean weakly t-divisible (resp. t-divisible) modules. By Proposition 1.8, a module M is weakly divisible if and only if $t(E(M)) = t(M)$.

Corollary 1.9. *Let E be an injective module. If E is $g.t$-full, then every essential submodule N of E is weakly divisible.*

Proposition 1.10. *In case t is a radical, for a module M and its submodule N, if M is a $g.t$-full module and $N \subseteq t(M)$, then M/N is $g.t$-full.*

Proof. Let L/N be an essential submodule of M/N. Then L is essential in M. By assumption, $t(L) = t(M)$ and M/L is in $T(t)$. Since $N \subseteq t(M)$, it follows that $t(M/N) = t(M)/N = t(L)/N \subseteq t(L/N)$ and hence $t(M/N) = t(L/N)$. Thus, M/N is $g.t$-full.

Proposition 1.11. *If a module M is semisimple, then M is $g.t$-full. Converse holds if M is t-torsion.*

Corollary 1.12. *Let M be a $g.t$-full module and N its essential submodule. If M/N is $g.t$-full, then it is semisimple.*

Theorem 1.13. *In case t is a radical, for any module M, $E(M)$ is $g.t$-full if and only if M is $g.t$-full and weakly divisible and $E(M)/M$ is in $T(t)$.*

Proof. It is sufficient to prove the "if" part. Let N be an essential submodule of $E(M)$. Since M is weakly divisible, $t(M) = t(E(M)) \supseteq t(N)$. On the other hand, since $N \cap M$ is essential in M and M is $g.t$-full,

$t(M) = t(N \cap M) \subseteq t(N)$. Thus, we have $t(N) = t(E(M))$. We now show that $E(M)/N$ is in $T(t)$. Consider the following diagram

$$\begin{array}{ccccccc} & & & & E(M)/M & & \\ & & & & \downarrow \text{canonical} & & \\ O \to (M+N)/N & \to & E(M)/N & \to & E(M)/(M+N) & \to & O \\ \downarrow \simeq & & & & \downarrow & & \\ M/(M \cap N) & & & & O & & \end{array}$$

where $E(M)/M$ and $M/(M \cap N)$ are in $T(t)$. Since $T(t)$ is closed under extensions, it follows that $E(M)/N$ is in $T(t)$.

2. Left exact preradicals t for which every module is $g.t$-full.

Proposition 2.1. *The following conditions are equivalent:*

1) *Every t-torsion module is semisimple.*
2) *Every cyclic t-torsion module is semisimple.*
3) *Every module is weakly divisible.*
4) *Every t-torsion module is weakly divisible.*
5) *$Soc \geq t$.*

Proof. 1) $\Rightarrow$ 2) is clear. 2) $\Rightarrow$ 1). Let M be a t-torsion module. We put $M = \sum_{m \in M} Rm$. Since M is in $T(t)$, each Rm is in $T(t)$. Hence by assumption, $Soc(Rm) = Rm$. Thus, M is semisimple. 1) $\Rightarrow$ 3). Let M be a module. Since t is left exact, $t(M)$ is essential in $t(E(M))$. By assumption, $t(E(M))$ is semisimple and hence $t(M) = t(E(M))$. Thus, M is weakly divisible. 3) $\Rightarrow$ 4) is trivial. 4) $\Rightarrow$ 5). Let M be a t-torsion module and N its submodule. Then N is in $T(t)$ and is weakly divisible by assumption. Therefore N is a direct summand of M. Thus, M is semisimple, which means that $t \leq Soc$. 5) $\Rightarrow$ 1). Let M be in $T(t)$. Then we have $M = t(M) \subseteq Soc(M)$. Hence, M is semisimple.

Remark. *In case t is a radical, each condition of the preceding proposition is also equivalent to*

6) *Every t-torsion module is divisible.*

Proof. 1) $\Rightarrow$ 6). Let M be in $T(t)$. Since t is a radical, $t(E(M)/t(E(M))) = O$, $t(E(M))$ is divisible by [5, Proposition 2.3]. Also since t is left exact, $M = t(M)$ is essential in $t(E(M))$. By assumption, $t(E(M))$

is semisimple and so $M = t(E(M))$. Hence M is divisible. 6) $\Rightarrow$ 1). Let M be in $T(t)$ and N its essential submodule. Then N is in $T(t)$ and is divisible. Hence it is a direct summand of M. Thus, $N = M$ and M is semisimple.

It is to be noted that in the above remark we can not remove the assumption that t is a radical, since if every t-torsion module is divisible then t must be a radical (cf. [6, Lemma 1.6]).

Proposition 2.2. *If a module M is g.t-full, then $Soc(M) \supseteq t(M)$. Moreover, R is g.t-full if and only if $t(R)$ is semisimple and $t \geq Z$.*

Proof. Let M be a $g.t$-full module and N its essential submodule. Then $t(M) = t(N) \subseteq N$. Since $Soc(M) = \cap \{ {}_RN \subseteq M \mid N$ is essential in $M \}$, $t(M) \subseteq Soc(M)$. Next assume R is $g.t$-full and let I be an essential left ideal of R, then $t(R/I) = R/I$. Thus I is in $L(t)$ and so $t \geq Z$. Conversely, if $Soc(R)$ contains $t(R)$ and $t \geq Z$, for each essential left ideal I of R. I contains $t(R)$and so $t(R) = I \cap t(R) = t(R)$. Therefore we see R is $g.t$-full.

We shall call a preradical r *cohereditary* if $r(M/N) = (r(M)+N)/N$ for every module M and its submodule N. As is well-known, this is a radical and $F(r)$ is closed under factor modules. An idempotent cohereditary preradical will be called a *cotorsion radical.*

Proposition 2.3. *Let r be a preradical. Then R is a semisimple ring if and only if R is g.r-full and r is cohereditary.*

Proof. It is sufficient to prove the "if" part. Let M be a module and let $x \in Z(M)$. Then the left annihilator $(O:x)$ of x in R is essential in R. Hence, $r((O:x)) = r(R)$ and $r(R/(O:x)) = R/(O:x)$. Since r is cohereditary, it follows that $R = r(R) + (O:x)$. Thus, we have $x = O$, from which we conclude that R has no proper essential left ideal, namely, R is semisimple.

Corollary 2.4. *If t is a cotorsion radical, then R is a semisimple ring if and only if R is a g.t-full module.*

A preradical s is said to be *of simple type* if $s = t_{\mathscr{S}}$ for some class $\mathscr{S}$ of simple modules, where $t_{\mathscr{S}}$ means the smallest one of those preradicals

r for which $r(S) = S$ for all $S \in \mathscr{S}$.

Theorem 2.5. *The following conditions are equivalent :*

1) *Every module is g.t-full.*
2) *Every injective module is g.t-full.*
3) *Every module is weakly divisible and* $t \geq Z$.
4) t *is of simple type and* $t \geq Z$.
5) $Z \geq t \geq Soc$.

Proof. 1) ⇒ 2) is clear. 2) ⇒ 3) is a direct consequence of Corollary 1.8 and [6, Lemma 1.10]. 3) ⇒ 4) follows from [7, Theorem 2.2]. 4) ⇒ 5) is clear. 5) ⇒ 1). Let M be a module and L its essential submodule. By assumption, $t \geq Z$. Hence, we have $t(M/L) = M/L$. Since $Soc(M) = Soc(L)$, $t(Soc(M)) = t(Soc(L))$ and so $t(M) \cap Soc(M) = t(L) \cap Soc(L)$. Therefore, $t(M) = t(L)$ by the fact that $t \leq Soc$. Thus, M is $g.t$-full.

In general, Soc is not larger than Z(for example $R = \mathbf{Z}$).

Example 2.6. Let R be the ring of 2×2 upper triangular matrices over a field K. Then $\mathscr{S}(Z) = \{R, Soc(R)\}$ and $\mathscr{S}(Soc) = \left\{R, \begin{pmatrix} K & K \\ O & O \end{pmatrix}, \begin{pmatrix} O & K \\ O & K \end{pmatrix}, \begin{pmatrix} O & K \\ O & O \end{pmatrix}\right\}$. Hence, $Soc \gneq Z$, and by Theorem 2.5, every module is $g.Z(Soc)$-full.

REFERENCES

[1] J. GOLAN : Torsion Theories, Pitman Monographs and Surveys in Pure and Applied Mathematics 29.

[2] O. GOLDMAN : Rings and modules of quotients, J. Algebra 13 (1969), 10–47.

[3] Y. KURATA : n-fold Torsion Theories in the category of modules, preprint.

[4] S. MORIMOTO : Weakly divisible and divisible modules, Tsukuba J. Math. 6 (1982), 195–200.

[5] S. MORIMOTO : Note on strongly M-injective modules, Math. J. Okayama Univ. 25 (1983), 165–171.

[6] S. MORIMOTO : On left exact preradicals, Math. J. Okayama Univ. 27 (1985), 87–95.

[7] M. SATO : On pseudo-cohereditary subtorsion theory and weakly divisible modules II, Memories of the Faculty of Liberal Art and Education Yamanashi University, 35 (1984), 19–24.

[8] K. SHIGENAGA : On the generalized t-full modules, Res. Rep. Ube Tech. Coll. 34 (1988), 31–33.

[9] B. STENSTRÖM : Rings of Quotients, Grundl. Math. Wiss. 217, Springer-Verlag, 1975.

S. MORIMOTO
HAGI KOEN GAKUEN HIGH SCHOOL
HIGASHITAMACHI, HAGI, YAMAGUCHI, JAPAN 758
K. SHIGENAGA
UBE TECHNICAL COLLEGE
TOKIWADAI, UBE, YAMAGUCHI, JAPAN 755

(*Received March 6, 1989*)