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ON SOME TYPE OF LEFT EXACT RADICALS

Dedicated to Professor Takashi Nagahara on his 60th birthday

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In his paper [7], V. S. Ramamurthi has studied the smallest left exact radical J^* larger than the Jacobson radical J. However if we take $\mathscr{C}$ as the class of cosingular modules, then J^* coincides with $t_{\mathscr{C}}$. In this note, first we shall study smallest left exact radical r^* larger than a preradical r and show that r^* can be described by various methods. Also we shall show that $(r_1 \wedge \cdots \wedge r_n)^* = r_1^* \wedge \cdots \wedge r_n^*$ (Theorem 1.8). Then we shall treat the largest left exact radical r_* smaller than a preradical r. Finally, we shall investigate a module M such that $(k_M)^*(M) = M$. In consequence, we can prove that every direct product M^Λ of copies of a module M which is non-singular and $(k_M)^*(M) = M$ has no nonzero injective submodule for any index set Λ (Theorem 2.9). Though we touch upon QF-3′ modules briefly in this paper, it has minutely been studied by Bican [1] and Kurata and Katayama [4].

Throughout this note R means a ring with identity and modules mean unitary left R-modules unless otherwise stated. We denote the category of modules by R-mod and the injective hull of $M \in R$-mod by $E(M)$. As for terminologies and basic properties concerning torsion theories and preradicals, we refer to [8]. For each preradical r, we denote the *r-torsion* (resp. *r-torsionfree*) class by $T(r)$ (resp. $F(r)$). Also the left linear topology corresponding to a left exact preradical r is denoted by $\mathscr{L}(r)$. Now for two preradicals r and s, we shall say that r *is larger than* s if $r(M) \supseteq s(M)$ for all modules M. For a preradical r, we put $\bar{r}(M) = \bigcap\{{}_RN \subseteq M \mid r(M/N) = O\}$, $\tilde{r}(M) = r(E(M)) \cap M$ and $\hat{r}(M) = \sum\{{}_RN \subseteq M \mid r(N) = N\}$ for each module M, where $O := \{0\}$. Then $\bar{r}$ (resp. $\tilde{r}$) is the smallest radical (resp. left exact preradical) larger than r and $\hat{r}$ is the largest idempotent preradical smaller than r.

1. We shall begin with the well-known lemma.

Lemma 1.1. *Let r and s be preradicals. Then the following statements hold.*

(1) *$F(r) = F(\bar{r})$ and $T(r) = T(\hat{r})$.*

$\wedge \bar{s}_2) = T(\overline{s_1 \wedge s_2})$. Hence $\overline{s_1 \wedge s_2} = \bar{s}_1 \wedge \bar{s}_2$ by Lemma 1.1(3).

In contrast with r^*, we now treat the largest left exact radical r_* smaller than a preradical r.

Proposition 1.9. [3, Corollary 3.13]. *Let r be a preradical for R-mod. We put $\mathscr{F} = \{E(X) \mid X \in F(r)\}$. Then $k_{\mathscr{F}}$ is the largest left exact radical smaller than $\bar{r}$.*

Proof. Since $k_{\mathscr{F}} = \wedge\{k_{E(X)} \mid X \in F(r)\} \leq \wedge\{k_X \mid X \in F(r)\} = k_{F(r)} = \bar{r}$, $k_{\mathscr{F}}$ is a left exact radical smaller than $\bar{r}$. Suppose that t is a left exact radical such that $t \leq \bar{r}$. Let M be an r-torsionfree module. Since $F(r) \subseteq F(t)$ and t is left exact, $E(M)$ is in $F(t)$, namely, $\mathscr{F} \subseteq F(t)$. Thus $k_{\mathscr{F}} \geq k_{F(t)} = t$, namely, $t = k_{\mathscr{F}}$.

For a preradical r, we denote by r_* the largest left exact radical smaller than r, if it exists.

Corollary 1.10. *Let r be a preradical for R-mod. Then*

(1) *If r is a radical, then r_* exists.*

(2) *r is a left exact radical if and only if there exists r_* and $r^* = r = r_*$ holds.*

The following example shows that if r is not a radical, then r_* need not exist in general.

Example 1.11. Let K be a field and let R be the ring of all 2×2 upper triangular matrices over K. Let r be the left exact preradical corresponding to the left linear topology having the smallest element

$$\begin{pmatrix} O & K \\ O & O \end{pmatrix}.$$

Also r_1 and r_2 mean the left exact radicals corresponding to the left Gabriel topologies having the smallest elements

$$\begin{pmatrix} K & K \\ O & O \end{pmatrix} \text{ and } \begin{pmatrix} O & K \\ O & K \end{pmatrix}$$

respectively. Then r is larger than r_1 and r_2 properly. If r_* exists, then r_* is larger than r_1 and r_2. Thus we obtain

$$\begin{pmatrix} O & K \\ O & O \end{pmatrix} \in \mathscr{L}(r_*) \text{ and so } \begin{pmatrix} O & K \\ O & O \end{pmatrix}^2 = O$$

which belongs to $\mathscr{L}(r_*)$. Hence $r_* = 1$ (identity). This is a contradiction.

Proposition 1.12. *Let r be a radical for R-mod. If each nonzero cyclic module in $T(r_*)$ has a nonzero factor in $F(r)$, then $r_* = O$. In particular, $J_* = O$, where J is the Jacobson radical.*

Proof. Suppose that there exists a nonzero module M such that $r_*(M) \neq O$. Then for any nonzero element x of $r_*(M)$, Rx is in $T(r_*)$. By hypothesis, there exists a proper submodule N of Rx such that Rx/N is in $F(r)$. On the other hand, Rx/N is in $T(r_*)$ and is in $F(r_*)$, since $r_* \leq r$. This is a contradiction. Now let $\mathscr{S}$ be a complete representative set of simple modules. We put $\mathscr{S}' = \{E(S) \mid S \in \mathscr{S}\}$. As is easily seen, $J = k_{\mathscr{S}}$ and $J_* = k_{\mathscr{S}'}$. Every nonzero cyclic module M has a maximal submodule X and M/X is in $F(J)$. Hence by the first half of the proposition, $J_* = O$.

Let $\mathscr{X}$ be a class of modules closed under injective hulls. We put $\mathscr{X}' = \{E(X) \mid X \in \mathscr{X}\}$. Since $\mathscr{X}'$ is a subclass of $\mathscr{X}$, $k_{\mathscr{X}'} \geq k_{\mathscr{X}}$. Also since $(k_{\mathscr{X}})_* = k_{\mathscr{X}'}$, $k_{\mathscr{X}'} = k_{\mathscr{X}}$. Thus $k_{\mathscr{X}}$ is left exact. If $\mathscr{X}$ is the class of modules with essential socles, then $k_{\mathscr{X}} = O$. In fact, since $Soc(E(M)) = Soc(M)$, $\mathscr{X}$ is closed under essential extensions. Also since $k_{\mathscr{X}} \leq J$ and $k_{\mathscr{X}} = (k_{\mathscr{X}})_*$, $k_{\mathscr{X}} \leq J_* = O$. Hence $k_{\mathscr{X}} = O$.

Proposition 1.13. *Let S be a simple module. Then S is injective if and only if $J(E(S)) = O$. In particular, J is left exact if and only if R is a left V-ring* [5, Proposition 5.3].

Proof. By [4, Corollary 2.6], every simple QF-3′ module is injective. Let S be a simple module such that $J(E(S)) = O$. We put $\mathscr{S}^{\beta} = \mathscr{S} - \{S_\beta\}$, where $\mathscr{S}$ is a complete representative set of simple modules and S_β is in $\mathscr{S}$ isomorphic to S. Since $k_{\mathscr{S}^\beta}(E(S)) \supseteq S$, $O = k_{\mathscr{S}}(E(S)) = k_{\mathscr{S}^\beta}(E(S)) \cap k_{S_\beta}(E(S)) \supseteq S \cap k_{S_\beta}(E(S))$. This implies $k_{S_\beta}(E(S)) = O$. Thus $k_S(E(S)) = O$, namely, S is QF-3′. Hence S is injective.

Let R be a left V-ring and a left semi-artinian ring. Then $Soc(M)$ is essential in M for all modules M. Also $k_M \leq k_{Soc(M)} = \wedge_{i \in I} k_{S_i}$ for some family $\{S_i\}_{i \in I}$ of simple modules. Since R is a left V-ring, S_i is injective for all $i \in I$. Thus $k_{Soc(M)}$ is left exact. Also since $E(Soc(M)) = E(M)$,

$k_{E(soc(M))} = k_{E(M)} \leq k_M \leq k_{soc(M)}$. Since $k_{soc(M)}$ is left exact, $k_{E(soc(M))} = k_{soc(M)}$, namely, k_M is left exact. Hence every module is QF-3' [6, Proposition 2].

Proposition 1.14. *Let $\{r_i\}_{i \in I}$ be a family of preradicals for R-mod. We put $\mathscr{F} = \bigcap \{F(r_i) \mid i \in I\}$. Then $(k_{\mathscr{F}})_*$ is the largest one of those left exact radical r for which $r(X) = 0$ for all $X \in \mathscr{F}$. Furthermore $(k_{\mathscr{F}})_* = (\overline{\sum_i r_i})_*$, where $(\sum_i r_i)(M) = \sum_i r_i(M)$ for each module M.*

Proof. Let X be a module. Then $X \in F(\sum_i r_i)$ if and only if $X \in F(r_i)$ for all $i \in I$, equivalently $X \in \bigcap \{F(r_i) \mid i \in I\} = \mathscr{F}$. Thus $F(\sum_i r_i) = \mathscr{F}$ and so $(\overline{\sum_i r_i}) = k_{\mathscr{F}}$. Therefore $(k_{\mathscr{F}})_* = (\overline{\sum_i r_i})_*$ is the largest left exact radical one of those left exact radical r such that $r(\mathscr{F}) = 0$.

Proposition 1.15. *Let $\{r_i\}_{i \in I}$ be a family of left exact preradicals. We put $\mathscr{F} = \bigcap \{F(r_i) \mid i \in I\}$. Then $k_{\mathscr{F}}$ is a left exact radical.*

Proof. Let r be a left exact preradical for R-mod. Then $\bar{r}$ is left exact if and only if $F(r)$ is closed under essential extensions. In fact, if $\bar{r}$ is left exact, then $F(\bar{r}) = F(r)$. Thus $r(E(X)) = 0$ for all $X \in F(r)$, namely, $F(r)$ is closed under essential extensions. Since each r_i is left exact, $F(r_i)$ is closed under essential extensions, and so is $\mathscr{F}$. Also since $\mathscr{F} = F(\sum_i r_i)$, $(\overline{\sum_i r_i}) = k_{\mathscr{F}}$ that is to say $k_{\mathscr{F}}$ is a left exact radical.

2. As above, k_M is the largest one of those preradical r for which $r(M) = 0$. In this section, we shall study $(k_M)^*$ for each module M and characterize those modules M for which $(k_M)^*(M) = M$.

Lemma 2.1. *Let $\mathscr{A} = \{M_\lambda\}_{\lambda \in \Lambda}$ be a family of modules. We put $M = \sum_{\lambda \in \Lambda} \oplus M_\lambda$ and $M' = \prod_{\lambda \in \Lambda} M_\lambda$. Then the following assertions hold.*

(1) $\wedge \{k_{M_\lambda} \mid \lambda \in \Lambda\} = k_M = k_{M'}$.

(2) $(k_M)^* = (k_{M'})^* \leq \wedge \{(k_{M_\lambda})^* \mid \lambda \in \Lambda\}$.

(3) $(k_A)^* = (k_{A^{(I)}})^* = (k_{A^I})^*$ *for all modules A and all index sets I.*

Proof. (1). Since $M_\lambda \subseteq M \subseteq M'$ for any $\lambda \in \Lambda$, $k_{M_\lambda} \geq k_M \geq k_{M'}$. Thus $\wedge \{k_{M_\lambda} \mid \lambda \in \Lambda\} \geq k_M \geq k_{M'}$. However $\wedge \{k_{M_\lambda} \mid \lambda \in \Lambda\} = k_{M'}$. Thus $\wedge \{k_{M_\lambda} \mid \lambda \in \Lambda\} = k_{M'} = k_M$. (2). $(k_M)^* = (k_{M'})^*$ and this equal to

$$(\wedge \{k_{M_\lambda} \mid \lambda \in \Lambda\})^* = \widetilde{\wedge \{k_{M_\lambda} \mid \lambda \in \Lambda\}}$$

$$\leq \overline{\wedge \{\tilde{k}_{M_\lambda} \mid \lambda \in \Lambda\}} \leq \wedge \{\bar{\tilde{k}}_{M_\lambda} \mid \lambda \in \Lambda\} = \wedge \{(k_{M_\lambda})^* \mid \lambda \in \Lambda\}.$$

(3) is clear.

Now we put $\mathcal{M}_1 = \{M \in R\text{-mod} \mid (k_M)^*(M) = O\}$, $\mathcal{M}_2 = \{M \in R\text{-mod} \mid (k_M)^*(M) = M\}$ and $\mathcal{M}_3 = \{M \in R\text{-mod} \mid O \neq (k_M)^*(M) \neq M\}$. As is easily seen, every injective module belongs to $\mathcal{M}_1$ and every noninjective simple module belongs to $\mathcal{M}_2$ by Proposition 1.13. Next let S_1 and S_2 be non-isomorphic simple noninjective modules (for example, $R = \mathbf{Z}$, $S_1 = \mathbf{Z}/2\mathbf{Z}$ and $S_2 = \mathbf{Z}/3\mathbf{Z}$). We put $M = E(S_1) \oplus S_2$. Then $(k_M)^*(M) = (k_M)^*(E(S_1)) \oplus (k_M)^*(S_2)$. By Theorem 1.8 and Lemma 2.1, $(k_M)^*(E(S_1)) = (k_{E(S_1)})^*(E(S_1)) \cap (k_{S_2})^*(S_2) = O$ and $(k_M)^*(S_2) = (k_{E(S_1)})^*(S_2) \cap (k_{S_2})^*(S_2) = k_{E(S_1)}(S_2) \cap (k_{S_2})^*(S_2) = k_{E(S_1)}(S_2) \cap S_2$. Assume that $Hom_R(S_2, E(S_1)) \neq O$. Then there exists a nonzero monomorphism $f \in Hom_R(S_2, E(S_1))$. Since $f(S_2) \cap S_1 \neq O$ and $f(S_2)$ is simple, $f(S_2) = S_1$, namely, $S_1 \simeq S_2$. This is a contradiction. Thus $k_{E(S_1)}(S_2) = S_2$. Hence $O \neq (k_M)^*(M) = S_2 \neq M$.

Corollary 2.2. [4, Proposition 2.2]. *Let $\{Q_\lambda\}_{\lambda \in \Lambda}$ be a family of QF-3' modules. We put $Q = \sum_{\lambda \in \Lambda} \oplus Q_\lambda$ and $Q' = \prod_{\lambda \in \Lambda} Q_\lambda$. Then both Q and Q' are QF-3'.*

Proof. By Lemma 2.1, $(k_Q)^* = (k_{Q'})^* \leq \wedge \{(k_{Q_\lambda})^* \mid \lambda \in \Lambda\} = \wedge \{k_{Q_\lambda} \mid \lambda \in \Lambda\} = k_Q = k_{Q'}$. Thus $k_Q = (k_Q)^*$ and $k_{Q'} = (k_{Q'})^*$.

In general, $k_{E(R)} \leq Z \leq G$ holds, where Z is the *singular torsion functor* and G is the *Goldie torsion functor*. If M is QF-3', faithful and nonsingular module, then k_M is left exact, $k_M \leq k_{E(R)}$ and $Z \leq k_M$. Thus $k_M \leq k_{E(R)} \leq Z \leq k_M$ and so $k_M = k_{E(R)} = Z$. Conversely, if $k_M = k_{E(R)} = Z$, then k_M is left exact, $k_M(R) = O$ and $Z(M) = O$. Hence M is QF-3', faithful and nonsingular. Thus we have

Proposition 2.3. *A module M is nonsingular, faithful and QF-3' if and only if $Z = k_{E(R)} = k_M$.*

Corollary 2.4. *For a ring R, the following conditions are equivalent:*

(i) *R is a left nonsingular ring.*

(ii) *There exists a faithful nonsingular module.*

(iii) *$Z = k_{E(R)}$.*

Proof. (i) ⇒ (ii) and (iii) ⇒ (i) are clear. (ii) ⇒ (iii). Let M be a faithful nonsingular module. Then so is $E(M)$. By Proposition 2.3,

$Z = k_{E(M)} = k_{E(R)}$.

Let ${}_RU$ be a module and S the endomorphism ring of ${}_RU$. For each module M, we put $M^* = Hom_R(M, {}_RU_S)$ and $M^{**} = Hom_S(M^*, {}_RU_S)$. There exists a natural mapping φ_M: $M \to M^{**}$ as $\varphi_M(m)(h) = h(m)$ for any $m \in M$ and any $h \in M^*$. As is well-known, $Ker(\varphi_M) = \{m \in M \mid h(m) = O$ for all $h \in M^*\} = k_U(M)$.

A module M is called *U-torsionless* if φ_M is a monomorphism, equivalently, $k_U(M) = O$. Clearly, M is QF-3' if and only if $E(M)$ is M-torsionless.

Proposition 2.5. *Let M be a faithful module. If E(M) is R-torsionless, then M is QF-3'.*

Proof. Since M is faithful, $k_M \leq k_R$. Also since $E(M)$ is R-torsionless, $k_R(E(M)) = O$, namely, $k_R \leq k_{E(M)}$. Thus $k_M \leq k_R \leq k_{E(M)} \leq k_M$ and so $k_M = k_{E(M)}$. Hence M is QF-3'.

However the converse is false. Take for example $R = \mathbf{Z}$ and $M = \mathbf{Q}$. Since ${}_{\mathbf{Z}}\mathbf{Q}$ is injective, ${}_{\mathbf{Z}}\mathbf{Q}$ is QF-3' and so M is faithful and QF-3'. But since $Hom_{\mathbf{Z}}(\mathbf{Q}, \mathbf{Z}) = O$, $k_R(M) = M$, namely, $E(M) = M$ is not R-torsionless.

Next we consider those modules M for which $(k_M)^*(M) = M$.

Proposition 2.6. *Let M be a module. Then the following assertions hold.*

(1) *If $(k_M)^*(M) = M$, then M has no nonzero injective submodule.*

(2) *$(k_M)^* = 1$ if and only if M^Λ has no nonzero injective submodule for any index set Λ.*

Proof. (1). Suppose that M has a nonzero injective submodule N. Then there exists a submodule X such that $M = N \oplus X$. Since $(k_{N\oplus X})^* = (k_N \wedge k_X)^* = (k_N)^* \wedge (k_X)^*$ and $(k_N)^*(M) = (k_N)^*(N \oplus X) = (k_N)^*(N) \oplus (k_N)^*(X) = (k_N)^*(X)$, $M = (k_M)^*(M) = k_X^*(M) \cap k_N^*(M) \subseteq X$, and so $M = X$ and $N = O$. This is a contradiction. (2). By the definition of k_M a module X belongs to $F(k_M)$ if and only if X can be embedded in M^Λ for some index set Λ. Let $\mathscr{E}'$ be the class of injective modules belonging to $F(k_M)$. Then $(k_M)^* = k_{\mathscr{E}'}$. Thus $(k_M)^* = 1$ if and only if $\mathscr{E}' = O$. The proof of (2) is completed.

If $R = \mathbb{Z}$ and $M = \mathbb{Z}/p\mathbb{Z}$, where p is a prime number, then M^{Λ} has no nonzero injective submodule for any index set Λ.

Corollary 2.7. *Let M be a module. If $(k_M)^*(M) = M$, then $M^{(\Lambda)}$ has no nonzero injective submodule for all index sets Λ.*

Proof. By Lemma 2.1 (3), $(k_{M^{(\Lambda)}})^*(M^{(\Lambda)}) = (k_M)^*(M^{(\Lambda)}) = ((k_M)^*(M))^{(\Lambda)} = M^{(\Lambda)}$. Thus $M^{(\Lambda)}$ has no have nonzero injective submodule by Proposition 2.6 (1).

Though the following lemma ([2, Lemma 0.2]) has been already known, we give here its torsion theoritic proof.

Lemma 2.8. *Let E be an injective module and M a nonsingular module. If $f \in Hom_R(E, M)$, then both $Im(f)$ and $Ker(f)$ are injective.*

Proof. Since $Ker(f)$ is essential in $E(Ker(f))$, $Z(E/Ker(f)) \supseteq Z(E(Ker(f))/Ker(f)) = E(Ker(f))/Ker(f)$. On the other hand, since $E/Ker(f) \simeq Im(f) \subseteq M$, $Z(E/Ker(f)) = 0$ and so $E(Ker(f))/Ker(f) = 0$. Therefore $E(Ker(f)) = Ker(f)$, namely, $Ker(f)$ is injective. Since $Ker(f)$ is a direct summand of E, there exists a submodule X of E such that $E = Ker(f) \oplus X$. Thus $X \simeq E/Ker(f) \simeq Im(f)$ and so $Im(f)$ is injective.

With regard to the converse of Proposition 2.6 (1), it seems to be difficult to prove the general case. For a nonsingular module M, we have

Theorem 2.9. *For a nonzero nonsingular module M, the following conditions are equivalent:*

(i) $(k_M)^*(M) = M$.
(ii) *M has no nonzero injective submodule.*
(iii) *$M^{(\Lambda)}$ has no nonzero injective submodule for all index sets Λ.*
(iv) *M^{Λ} has no nonzero injective submodule for all index sets Λ.*
(v) $(k_M)^* = 1$.

Proof. (i)$\Rightarrow$(iii) follows from Corollary 2.7. The equivalence of (iv) and (v) was proved in Proposition 2.6 (2). Since (iii)$\Rightarrow$(ii) and (v)$\Rightarrow$(i) are clear, we may prove (ii)$\Rightarrow$(iv). Assume that M^{Λ} has a nonzero injective submodule E for some index set Λ. Thus there exists a nonzero homomorphism f from E to M. By Lemma 2.8, $Im(f)$ is nonzero injective, namely, M has a nonzero injective submodule. This is a contradiction.

Example 2.10. Let R be the ring of 2×2 upper triangular matrices over a field K. We put

$$M=\begin{pmatrix} K & O \\ O & O \end{pmatrix}.$$

Then M is a simple projective module and so M is nonsingular. Clearly M is not injective, namely, $(k_M)^*(M)=M$.

Note that if $R=\mathbb{Z}$ and $M=\mathbb{Z}/p\mathbb{Z}$, as above. Then M is singular, noninjective and simple, but satisfies the equivalent condition of Theorem 2.9.

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