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QUOTIENT RINGS OF Φ-TRIVIAL EXTENSIONS

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Let R be a ring with identity and M an (R, R)-bimodule. In his paper [3], Kitamura has shown that every right quotient ring of the trivial extension of R by M is a trivial extension of a right quotient ring of R by a suitable bimodule if ${}_RM$ is flat and is finitely generated by elements each of which commutes with every element of R.

The purpose of the present paper is to extend this result to the corresponding one for Φ-trivial extensions. Let Λ be the Φ-trivial extension of R by an (R, R)-bimodule M with pairing Φ. For each R-module U_R, $V = U \oplus M^*$ can be seen as a right Λ-module, where M^* is the dual of M relative to U. In Sectiuon 2 it is shown that $\mathrm{Biend}(V_\Lambda) \simeq \mathrm{Biend}(U_R) \times_\theta M^{**}$ if M^*_R is U-reflexive (Theorem 2.1). Under certain assumptions, in Section 3, we shall observe the injective hull of any right Λ-module (Theorem 3.2) and then determine the right quotient ring of Λ_Λ as $Q(\Lambda_\Lambda) \simeq Q(R_R) \times_{\theta''} Q(M_R)$ (Theorem 3.3).

Throughout this paper, R will denote a ring with identity. All modules are unitary and module homomorphisms are written on the side opposite to the scalars. We shall refer to [1] for the notations and terminologies concerning the ring theory.

1. Let M be an (R, R)-bimodule with pairing $\Phi = [\ ,\ ] : M \otimes_R M \to R$, i.e. an (R, R)-bilinear map satisfying $m[m', m''] = [m, m']m''$ for all m, m' and m'' in M. The Φ-trivial extension $\Lambda = R \times_\Phi M$ of R by M is a ring whose underlying set is the Cartesian product $R \times M$ with addition componentwise and multiplication given by

$$(a, m)\cdot(a', m') = (aa' + [m, m'], ma' + am').$$

For an R-module U_R, $\mathrm{Hom}_R(\Lambda, U)$ is canonically Z-isomorphic to $U \oplus M^*$, where $M^* = \mathrm{Hom}_R(M, U)$ is the U-dual of M_R. Using this isomorphism we can regard $U \oplus M^*$ as a right Λ-module. The operation of Λ on $U \oplus M^*$ is given by

$$(u, f)\cdot(a, m) = (ua + f(m), fa + \varphi(u \otimes m))$$

for (u, f) in $U \oplus M^*$ and (a, m) in Λ, where φ: $U \otimes_R M \to M^*$ is the right R-homomorphism defined by $\varphi(u \otimes m)(m') = u[m, m']$ for m, m' in M and u

in U. We denote this right Λ-module by V.

Let $S = \mathrm{End}(U_R)$ and $N = \mathrm{Hom}_R(M^*, U)$. Then $\mathrm{End}(V_\Lambda) \simeq \mathrm{Hom}_\Lambda(V, \mathrm{Hom}_R(\Lambda, U)) \simeq \mathrm{Hom}_R(V, U) \simeq S \oplus N$ and the composite isomorphism $\mathrm{End}(V_\Lambda) \to S \oplus N$ is given by $\alpha \to (p_1\alpha i_1, p_1\alpha i_2)$, where i_K and p_k denote injections and projections associated with the direct sum decomposition of $V = U \oplus M^*$, respectively. We denote this isomorphism by τ.

We shall define a pairing $N \otimes_S N \to S$ through which $S \oplus N$ becomes a ring and τ is a ring isomorphism. To this end, first we show the following

Lemma 1.1. *For every $\alpha \in \mathrm{End}(V_\Lambda)$, $f \in M^*$ and $h \in N$,*

(1) $$p_2\alpha i_1 = \mathrm{Hom}(M, p_1\alpha i_2) \circ \varphi',$$

where $\varphi' : U \to \mathrm{Hom}_R(M, M^*)$ is the (S, R)-homomorphism given by $\varphi'(u)(m) = \varphi(u \otimes m)$.

(2) $$p_2\alpha i_2 = \mathrm{Hom}(M, p_1\alpha i_1).$$

(3) $$h \cdot p_1\alpha i_1 = h \circ p_2\alpha i_2.$$

Proof. (1) Let $\alpha i_1(u) = (u', f)$ for some u' in U and f in M^*. Then for each m in M $((p_2\alpha i_1)(u))(m) = f(m)$. On the other hand, $(p_1\alpha i_2 \circ \varphi'(u))(m) = (p_1\alpha i_2)(\varphi(u \otimes m)) = p_1\alpha((u, 0)\cdot(0, m)) = p_1((u', f)\cdot(0, m)) = f(m)$.

(2) Let $\alpha i_2(f) = (u, f')$ for some u in U and f' in M^*. Then for each m in M $((p_2\alpha i_2)(f))(m) = f'(m)$, while $(p_1\alpha i_1 \cdot f)(m) = p_1\alpha((0, f)\cdot(0, m)) = p_1((u, f')\cdot(0, m)) = f'(m)$.

(3) For each f in M^*, $(h \cdot p_1\alpha i_1)(f) = h(p_1\alpha i_1 \cdot f) = h((p_2\alpha i_2)(f)) = (h \circ p_2\alpha i_2)(f)$ by (2).

Now N is an (S, S)-bimodule and we define a mapping from $N \times N$ to S via $(p_1\alpha i_2, p_1\beta i_2) \to p_1\alpha i_2 \circ p_2\beta i_1$, where α and β are in $\mathrm{End}(V_\Lambda)$. This is well-defined by Lemma 1.1 (1) and induces a pairing $\Psi = \langle\ ,\ \rangle : N \otimes_S N \to S$ given by $\langle p_1\alpha i_2, p_1\beta i_2 \rangle = p_1\alpha i_2 \circ p_2\beta i_1$, i.e. for each h, h' in N and u in U, $\langle h, h' \rangle(u) = h(h' \circ \varphi'(u))$ again by Lemma 1.1(1). Therefore, $S \oplus N$ becomes the Ψ-trivial extension $\Gamma = S \times_\Psi N$ of S by N and further by Lemma 1.1 (3) τ is a ring isomorphism between $\mathrm{End}(V_\Lambda)$ and Γ. Thus, we obtain

Theorem 1.2. $\mathrm{End}(V_\Lambda)$ *is isomorphic to Γ as rings via τ.*

It follows from this theorem that V can be regarded naturally as a left Γ-module by making use of τ. The operation of Γ on V is given by

$$(s, h)\cdot(u, f) = \tau^{-1}(s, h)((u, f))$$

$$= (s(u)+h(f),\ sf+h\circ\varphi'(u))$$

for (u, f) in V and (s, h) in Γ.

Recall that S is a ring with identity, N is an (S, S)-bimodule with pairing $\Psi = \langle\ ,\ \rangle : N \otimes_S N \to S$ and $\Gamma = S \times_\Psi N$ is the Ψ-trivial extension of S by N. Replacing R, M and Λ with S, N and Γ, respectively, we have just the same situation as above. Therefore, for the left S-module U, $W = U \oplus N^*$, where $N^* = \mathrm{Hom}_S(N, U)$, becomes a left Γ-module with the operation of Γ given by

$$(s, h)\cdot(u, k) = (s(u)+(h)k,\ sk+\psi(h\otimes u))$$

for (s, h) in Γ and (u, k) in W. Here the mapping ψ: $N \otimes_S U \to N^*$ is defined by $(h')\psi(h\otimes u) = \langle h', h\rangle u$ for h' in N and is an (S, R)-homomorphism. Note that ψ coincides with the composition map of $N \otimes_S U \to M^*$ given by $h\otimes u \to h\circ\varphi'(u)$ with the evaluation map $\sigma_{M^*}: M^* \to N^*$ of M^*_R.

Now let $T = \mathrm{End}({}_SU)$. Then U is a right T-module and N^* is an (S, T)-bimodule. Hence, $L = \mathrm{Hom}_S(N^*, U)$ has a (T, T)-bimodule structure. Replacing S and N with T and L respectively, by the same way as above, we can define a pairing $\Omega = |\ ,\ | : L\otimes_T L \to T$ and the Ω-trivial extension $\Delta = T\times_\Omega L$ of T by L. The pairing Ω is $(u)|k, k'| = (\psi'(u)\circ k)k'$ for k, k' in L and u in U, where ψ': $U \to \mathrm{Hom}_S(N, N^*)$ is the (S, T)-homomorphism defined by $(h)\psi'(u) = \psi(h\otimes u)$ for h in N. Using Theorem 1.2, we see that

$$\mathrm{End}({}_\Gamma W) \simeq \Delta.$$

2. In this section we shall assume that M^*_R is U-reflexive. Then the evaluation map $\sigma = \sigma_{M^*}$ is an (S, R)-isomorphism and hence the mapping $U\oplus\sigma$: $V\to W$ is a Γ-isomorphism and induces a ring isomorphism

$$\mathrm{End}({}_\Gamma V) \simeq \mathrm{End}({}_\Gamma W).$$

Using σ we may also regard M^* as a right T-module, i.e. for $t\in T$ and $f\in M^*$ define ft to be $ft = ((f)\sigma\circ t)\sigma^{-1}$. Then M^* is an (S, T)-bimodule, σ is an (S, T)-isomorphism and $M^{**} = \mathrm{Hom}_S(M^*, U)$, the double dual of M_R, is a (T, T)-bimodule. Hence the mapping $\mathrm{Hom}(\sigma, U): L\to M^{**}$ is a (T, T)-isomorphism and yields a pairing $\Theta: M^{**}\otimes_T M^{**}\to T$ such that $\Omega = \Theta\circ\mathrm{Hom}(\sigma, U)\otimes\mathrm{Hom}(\sigma, U)$ and a ring isomorphism $1\times\mathrm{Hom}(\sigma, U): \Delta\to T\times_\Theta M^{**}$. Thus, we obtain

Theorem 2.1. *Assume that $M^*{}_R$ is U-reflexive. Then*

$$\mathrm{End}({}_{\Gamma}V) \simeq T \times_{\theta} M^{**}$$

as rings, i.e.

$$\mathrm{Biend}(V_{\Lambda}) \simeq \mathrm{Biend}(U_R) \times_{\theta} M^{**}.$$

The following corollary follows from [6, Theorem 1.4].

Corollary 2.2. *Let $U = E(M_R)$ and assume that M_R^* is $E(M_R)$-reflexive and ${}_RM$ is faithful. Then*

$$\mathrm{Q}_{\max}(\Lambda_{\Lambda}) \simeq \mathrm{Biend}(\mathrm{E}(M_R)) \times_{\theta} M^{**}.$$

As is easily seen, if R' is a ring with identity such that $R \simeq^{f} R'$ as rings, then for an (R, R)-bimodule M with pairing Θ: $M \otimes_R M \to R$, we can regard M naturally as an (R', R')-bimodule via f and find a pairing Θ': $M \otimes_{R'} M \to R'$ such that $R \times_{\theta} M \simeq R' \times_{\theta'} M$ as rings. Hence, by [6, Theorem 1.3] we have

Corollary 2.3. *Let $U = \mathrm{E}(R_R)$ and assume that $M^*{}_R$ is $E(R_R)$-reflexive and Φ is right non-degenerate. Then*

$$\mathrm{Q}_{\max}(\Lambda_{\Lambda}) \simeq \mathrm{Q}_{\max}(R_R) \times_{\theta'} M^{**}.$$

It is easily verified that the isomorphism in Theorem 2.1 induces a commutative diagram

$$\begin{array}{ccc} & \Lambda = R \times_{\Phi} M & \\ \swarrow^{\rho_{\Lambda}} & & \searrow^{\rho_R \times \sigma_M} \\ \mathrm{Biend}(V_{\Lambda}) & \simeq & \mathrm{Biend}(U_R) \times_{\theta} M^{**} \end{array}$$

where ρ_{Λ} and ρ_R are right multiplications of elements of Λ and R, respectively and σ_M is the evaluation map $M \to M^{**}$. Thus, we have

Corollary 2.4. *Assume that M_R^* is U-reflexive. Then V_{Λ} is (faithful and) balanced if and only if U_R is (faithful and) balanced and σ_M is (injective and) surjective.*

We can apply this corollary to Corollaries 2.2 and 2.3 and obtain that, for example, if M_R^* is $\mathrm{E}(R_R)$-reflexive and Φ is right non-degenerate, then Λ

is isomorphic to $Q_{max}(\Lambda_\Lambda)$ via ρ_Λ if and only if R is isomorphic to $Q_{max}(R_R)$ via ρ_R and M_R is $E(R_R)$-reflexive.

As an application of Theorem 1.2 we can give a criterion for $Q_{max}(\Lambda_\Lambda)$ being right self-injective. For example, if ${}_RM$ is faithful, then $Q_{max}(\Lambda_\Lambda)$ is right self-injective if and only if (1) $\mathrm{Hom}_R(M, E(M_R))$ is a free left S-module with a basis ν, the inclusion map $M \to E(M_R)$, and (2) ${}_SN$ is isomorphic to ${}_SE(M_R)$ via $(\nu)\sigma_{M^*}$. This result can be seen as a generalization of [3, Proposition 6] and is easily obtained using [4, Section 4.3, Proposition 3].

3. Let $\Lambda = R \times_\Phi M$ be the Φ-trivial extension of R by M as above and V_Λ any right Λ-module. Then since $\mathrm{Im}\,\Phi \times M$ is an ideal of Λ, the left annihilator $\ell_V(\mathrm{Im}\,\Phi \times M)$ of $\mathrm{Im}\,\Phi \times M$ in V is a Λ-submodule of V. We may regard V and $\ell_V(\mathrm{Im}\,\Phi \times M)$ naturally as right R-modules. Let $U = E(\ell_V(\mathrm{Im}\,\Phi \times M)_R)$ and put $E(V_R) = U \oplus U'$ for some R-submodule U' of $E(V_R)$. Using the projection map p: $E(V_R) \to U$, define a right Λ-homomorphism ξ: $V \to U \oplus M^*$ as $\xi(v) = (p(v), m \to p(v(0, m)))$ for v in V and m in M.

It is to be noted that $\ell_V(0 \times M)$ and $\ell_V(\mathrm{Im}\,\Phi)$ are both Λ-submodules of V and

$$\ell_V(\mathrm{Im}\,\Phi \times M) = \ell_V(0 \times M) \leq \ell_V(\mathrm{Im}\,\Phi) \leq V,$$

since $([m, m'], 0) = (0, m)\cdot(0, m')$ for $m, m' \in M$. Furthermore, $\ell_V(0 \times M)_\Lambda$ is essential in $\ell_V(\mathrm{Im}\,\Phi)_\Lambda$ and $\ell_U(\mathrm{Im}\,\Phi)_R$ is also essential in U_R. Using these facts we shall prove

Lemma 3.1. *The following conditions are equivalent*:

(1) *ξ is a monomorphism.*

(2) *$\ell_V(\mathrm{Im}\,\Phi \times M)_\Lambda$ is essential in V_Λ.*

(3) *$\ell_V(\mathrm{Im}\,\Phi)_\Lambda$ is essential in V_Λ.*

If this is the case, ξ becomes an essential monomorphism.

Proof. (2) $\to$ (1). Assume (2). Let $\xi(v) = 0$. Then $p(v) = 0$ and so $\mathrm{Ker}(\xi) \leq U'$. Since ξ is a Λ-homomorphism, it follows that $v\Lambda \leq \mathrm{Ker}(\xi)$ and hence $v\Lambda \cap \ell_V(\mathrm{Im}\,\Phi \times M) \leq U' \cap \ell_V(\mathrm{Im}\,\Phi \times M) = 0$. By assumption $v\Lambda = 0$ and hence $v = 0$.

Now we shall show that $\xi(V)_\Lambda$ is essential in $(U \oplus M^*)_\Lambda$. To this end take (u, f) ($\neq 0$) in $U \oplus M^*$. If $f = 0$, then $u \neq 0$ and hence there exists an a in R such that $0 \neq ua \in \ell_V(\mathrm{Im}\,\Phi \times M)$. In this case, $(u, f)\cdot(a, 0) = (ua, 0) = \xi(ua)$. If $u = 0$, then there is an m in M such that $f(m) \neq 0$. Hence we can find an a in R such that $0 \neq f(m)\cdot a \in \ell_V(\mathrm{Im}\,\Phi \times M)$. In this case, $(u, f)\cdot(0, ma) = (f(ma), 0) = \xi(f(ma))$.

Next suppose that $u \neq 0$ and $f \neq 0$. Then there exists an a in R such that $0 \neq ua \in \ell_V(\mathrm{Im}\,\Phi \times M)$ and $(u, f)\cdot(a, 0) = (ua, fa)$. In case $fa = 0$, we have $(u, f)\cdot(a, 0) = (ua, 0) = \xi(ua)$. If $fa \neq 0$, there exists an m in M for which $f(am) = (fa)(m) \neq 0$. We can find an a' in R such that $0 \neq f(am)\cdot a' \in \ell_V(\mathrm{Im}\,\Phi \times M)$. We then have $(u, f)\cdot(0, ama') = (ua, fa)\cdot(0, ma') = (f(ama'), 0) = \xi(f(ama'))$.

(1) → (2). Using the fact that $\ell_U(\mathrm{Im}\,\Phi)_R$ is essential in U_R, it is easy to see that $\ell_V(\mathrm{Im}\,\Phi \times M)_\Lambda$ is essential in V_Λ by a similar way as above.

The equivalence of (2) and (3) is trivial.

The following theorem characterizes injective modules over Λ and can be seen as a generalization of [6, Theorem 2.4].

Theorem 3.2. *For any right Λ-module V there is an injective right R-module U such that*

$$\mathrm{E}(V_\Lambda) \simeq U \oplus M^*$$

as right Λ-modules, whenever ξ is a monomorphism.

Proof. This follows from Lemma 3.1 and the fact that for any injective right R-module X, $X \oplus \mathrm{Hom}_R(M, X)$ is isomorphic to $\mathrm{Hom}_R(\Lambda, X)$ over Λ and hence is injective over Λ [1, Exercise (19.14)].

As is well-known, every hereditary torsion theory for mod-R is cogenerated by a certain injective R-module E_R. We shall call it simply the E-torsion theory.

Assuming that ξ is a monomorphism, we now discuss the problem of how to determine the quotient ring of the Φ-trivial extension Λ of R by M. Following Morita [5], every right quotient ring of Λ is isomorphic to the biendomorphism ring of a finitely cogenerating, injective right Λ-module.

So let V_Λ be a finitely cogenerating, injective right Λ-module. This means that V is injective over Λ and is finitely generated over $\mathrm{End}(V_\Lambda)$. Theorem 3.2 then implies that $V \simeq U \oplus M^*$ as right Λ-modules, for some injective right R-module U.

First assume that M_R^* is U-reflexive. Then by Theorem 2.1 $\mathrm{Biend}(V_\Lambda)$

$\simeq$ Biend$(U_R) \times_\theta M^{**}$ as rings. Now let $S = \text{End}(U_R)$ and $N = \text{Hom}_R(M^*, U)$. If we assume further that ${}_SN$ is finitely generated, then U_R is finitely cogenerating, since V_Λ is finitely cogenerating. Hence, there is a ring isomorphism Biend$(U_R) \to \text{Q}(R_R)$ over R, where $\text{Q}(R_R)$ denotes the right quotient ring of R with respect to the U_R-torsion theory. As we have remarked in Section 2, we can ragard M^{**} naturally a $(\text{Q}(R_R), \text{Q}(R_R))$-bimodule and find a pairing Θ': $M^{**} \otimes_{\text{Q}(R_R)} M^{**} \to \text{Q}(R_R)$ such that Biend$(U_R) \times_\theta M^{**} \simeq \text{Q}(R_R) \times_{\theta'} M^{**}$ as rings. Thus, we have

Theorem 3.3. *Let* $\text{Q}(\Lambda_\Lambda)$ *be any right quotient ring of* Λ, V *an associated finitely cogenerating, injective right* Λ*-module and* $U = \text{E}(\ell_V(\text{Im}\,\Phi \times M)_R)$. *Assume that* ξ *is a monomorphism and that* M_R^* *is* U*-reflexive. Then we have*

(1) $$\text{Q}(\Lambda_\Lambda) \simeq \text{Biend}(U_R) \times_\theta M^{**}.$$

(2) *If* ${}_SN$ *is finitely generated, then as rings*

$$\text{Q}(\Lambda_\Lambda) \simeq \text{Q}(R_R) \times_{\theta'} M^{**}.$$

(3) *If, in addition,* ${}_SM^*$ *is finitely generated, then*

$$\text{Q}(\Lambda_\Lambda) \simeq \text{Q}(R_R) \times_{\theta''} \text{Q}(M_R)$$

as rings, where $\text{Q}(M_R)$ *denotes the module of quotients of* M_R *with respect to the* U_R*-torsion theory.*

Proof. We may prove only (3). Suppose in addition that ${}_SM^*$ is finitely generated. Then by [2, Theorem 1.2] there exists an R-isomorphism k: $M^{**} \to \text{Q}(M_R)$ over M. Using this isomorphism, we can regard $\text{Q}(M_R)$ as a $(\text{Q}(R_R), \text{Q}(R_R))$-bimodule and define a pairing Θ'': $\text{Q}(M_R) \otimes_{\text{Q}(R_R)} \text{Q}(M_R) \to \text{Q}(R_R)$ such that $\Theta' = \Theta'' \circ k \otimes k$. Then we have a ring isomorphism $1 \times k$: $\text{Q}(R_R) \times_{\theta'} M^{**} \to \text{Q}(R_R) \times_{\theta''} \text{Q}(M_R)$ and thus $\text{Q}(\Lambda_\Lambda) \simeq \text{Q}(R_R) \times_{\theta''} \text{Q}(M_R)$.

As is easily seen, in case U_R is injective the condition that ${}_SM^*$ is finitely generated is equivalent to that U_R cogenerates M_R finitely and is always true if $\Phi = 0$ and V_Λ is finitely cogenerating as was pointed out by [3, Proposition 1].

Example 3.4. Let M be an (R, R)-bimodule and U_R an R-module. Assume that there exists a split exact sequence of right R-modules $0 \to M^* \to U^n$ for some $n > 0$. Then M_R^* is U-reflexive, since U_R itself is U-

reflexive and the class of U-reflexive modules is closed under direct summands and finite direct sums, and further $S^n \to N \to 0$ is exact. Hence ${}_SN$ is finitely generated. Moreover, if Λ is the trivial extension of R by M and V_Λ is finitely cogenerating, then ${}_SM^*$ is finitely generated, as we have remarked above. In this case ξ is a monomorphism by Lemma 3.1 and all of the pairings are zero. Thus we have

$$Q(\Lambda_\Lambda) \simeq Q(R_R) \ltimes Q(M_R),$$

by Theorem 3.3. This is a detailed form of [3, Theorem 4].

Example 3.5. Let ${}_RM_R = {}_RR_R$. Then, for every finitely cogenerating injective R-module U_R, $M_R^* \simeq U_R$, ${}_SN \simeq {}_SS$ and ${}_SM^* \simeq {}_SU$. Hence, for every Φ-trivial extension Λ of R by ${}_RR_R$, we have by Theorem 3.3

$$Q(\Lambda_\Lambda) \simeq Q(R_R) \times_{\theta''} Q(R_R)$$

as rings, whenever ξ is a monomorphism.

Example 3.6. Let $U_R = E(R_R)$ and let $\xi: \Lambda \to U \oplus M^*$ be the Λ-homomorphism defined by $\xi(a, m) = (a, m' \to [m, m'])$. Then ξ is a monomorphism if and only if Φ is right non-degenerate. Hence, assuming that Φ is right non-degenerate and M_R^* is U-reflexive, we have by Corollary 2.3

$$Q_{\max}(\Lambda_\Lambda) \simeq Q_{\max}(R_R) \times_{\theta'} M^{**}.$$

If we assume further that ${}_SM^*$ is finitely generated, then by a similar way as in Theorem 3.3 we have

$$Q_{\max}(\Lambda_\Lambda) \simeq Q_{\max}(R_R) \times_{\theta''} Q(M_R).$$

If, in particular, we assume that $M = {}_RR_R$ and Φ is given by the multiplication in R, then we have

$$Q_{\max}(\Lambda_\Lambda) \simeq Q_{\max}(R_R) \times_{\theta''} Q_{\max}(R_R)$$

without any restriction.

After completed this paper, we have found that there are some overlaps, for example, Theorems 1.2 and 3.2, with Eduardo Garcia-Herreros Mantilla: Semitriviale Erweiterungen und generalisierte Matrizenringe, München, 1986 (Algebra Berichte 54).

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