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ON HARADA RINGS III

Dedicated to Professor Teruo Kanzaki on his 60th birthday

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In [2], we showed that left Harada (abbreviated H-) rings can be represented as suitable generalized matrix rings. In the present paper, using this result, we shall further show that all left H-rings can be constructed by suitable extension rings of QF-rings and their factors. As a result, we see that left H-rings are (left and) right artinian.

Preliminaries. Throughout this paper, rings R considered are associative rings with identity and all R-modules are unitary. The notation M_R (resp. $_RM$) is used to stress that M is a right (resp. left) R-module. For an R-module M, $J(M)$ and $S(M)$ denote its Jacobson radical and socle, respectively, and $\{J_i(M)\}$ and $\{S_i(M)\}$ denote its descending Loewy chain and ascending Loewy chain, respectively.

For R-modules M and N, for the sake of convenience, we put $(M, N) = \mathrm{Hom}_R(M, N)$ and in particular, put $(e, f) = (eR, fR) = \mathrm{Hom}_R(eR, fR)$.

Now, in what follows, we assume that R is a basic left H-ring and E a complete set of orthogonal primitive idempotents of R. Namely, R is a basic left artinian ring and E is arranged as

$$E = \{e_{11}, \dots, e_{1n(1)}, \dots, e_{m1}, \dots, e_{mn(m)}\}$$

for which

1) each $e_{i1}R_R$ is injective,

2) there exists an isomorphism from $e_{ik}R_R$ to $e_{i,k-1}J(R)_R \simeq J(e_{i,k-1}R_R)$ for $1 \leqq i \leqq m$ and $2 \leqq k \leqq n(i)$.

we represent R as

$$R = \begin{pmatrix} (e_{11}, e_{11}) & \cdots & (e_{mn(m)}, e_{11}) \\ & \cdots & \\ (e_{11}, e_{mn(m)}) & \cdots & (e_{mn(m)}, e_{mn(m)}) \end{pmatrix} = \begin{pmatrix} e_{11}Re_{11} & \cdots & e_{11}Re_{mn(m)} \\ & \cdots & \\ e_{mn(m)}Re_{11} & \cdots & e_{mn(m)}Re_{mn(m)} \end{pmatrix}.$$

The following properties hold on R ([2]):

a) Each $S(e_{ij}R_R)_R$ is simple,

$$S(e_{i1}R_R)_R \simeq \cdots \simeq S(e_{in(i)}R)_R,$$
$$S(e_{ij}R_R)_R \not\simeq S(e_{kt}R)_R \text{ if } i \neq k.$$

b) For each e_{i1}, there exists a unique $g_i \in E$ such that $(e_{i1}R_R\,;Rg_i)$ is an injective pair, i.e., $g_iR_R/J(g_iR_R)_R \simeq S(e_{i1}R_R)_R$ and ${}_RRe_{i1}/J({}_RRe_{i1}) \simeq {}_RS({}_RRg_i)$. Then ${}_RRg_i$ is injective, and

$$S_k({}_RRg_i) = S(e_{i1}R_R)+\cdots+S(e_{ik}R_R)$$

for $1 \leqq i \leqq m$, $1 \leqq k \leqq n(i)$. So, $S_k(Rg_i)$ is a two-sided ideal. In particular, $S({}_RRg_i) = S(e_{i1}R_R)$ is a simple ideal. In the matrix representation,

$$S_k({}_RRg_i) = \begin{pmatrix} & 0 & \\ & \vdots & \\ & 0 & \\ & X_1 & \\ 0 & \vdots & 0 \\ & X_k & \\ & 0 & \\ & \vdots & \\ & 0 & \end{pmatrix}$$

where $X_j = S({}_{e_{ij}Re_{ij}}e_{ij}Rg_i) = S(e_{ij}Rg_{i\,g_iRg_i})$ for $1 \leq j \leq k$.

Here we define two mappings:

$$\sigma: \{1, \ldots, m\} \to \{1, \ldots, m\}$$
$$\rho: \{1, \ldots, m\} \to \{1, \ldots, n(1)\} \cup \cdots \cup \{1, \ldots, n(m)\}$$

by the rule $\sigma(i) = k$ and $\rho(i) = t$ if $g_i = e_{kt}$; namely $(e_{i1}R_R\,;Re_{\sigma(i)\rho(i)})$ is an injective pair. We note that $\{\sigma(1), \ldots, \sigma(m)\} \subseteq \{1, \ldots, m\}$ and $\rho(i) \leq n(\sigma(i))$.

Definition. We say that R is a left H-ring of type $(*)$ if $\{\sigma(1), \ldots, \sigma(n)\}$ is a permutation of $\{1, \ldots, n\}$ and $\rho(i) = n(\sigma(i))$ for $i = 1, \ldots, n$.

We define R_{ij} by putting

$$R_{ij} = \begin{pmatrix} (e_{j1}, e_{i1}) & \cdots & (e_{jn(j)}, e_{i1}) \\ & \cdots & \\ (e_{j1}, e_{in(i)}) & \cdots & (e_{jn(j)}, e_{in(i)}) \end{pmatrix}$$

$$= \begin{pmatrix} e_{i1}Re_{j1} & \cdots & e_{i1}Re_{jn(j)} \\ & \cdots & \\ e_{in(i)}Re_{j1} & \cdots & e_{in(i)}Re_{jn(j)} \end{pmatrix} .$$

Then

$$R = \begin{pmatrix} R_{11} & \cdots & R_{1m} \\ & \cdots & \\ R_{m1} & \cdots & R_{mm} \end{pmatrix} .$$

Notation. For the sake of convenience, we put $e_i = e_{i1}$, $A_{ij} = e_iRe_j$ $(i \neq j)$ and $Q_i = e_iRe_i$ for $1 \leqq i \leqq m$ and $1 \leqq j \leqq m$.

Now, we define $P_{ik,jt}$ corresponding to $e_{ik}Re_{jt}$ as follows:

$$P_{ik,jt} = \begin{cases} A_{ij} & (i \neq j) \\ Q_i & (i = j,\ k \leq t) \\ J(Q_i) & (i = j,\ k > t) \end{cases}$$

and put

$$P_{ij} = \begin{pmatrix} P_{i1,j1} & \cdots & P_{i1,jn(j)} \\ & \cdots & \\ P_{in(i),j1} & \cdots & P_{in(i),jn(j)} \end{pmatrix} .$$

Namely, when $i \neq j$,

$$P_{ij} = \begin{pmatrix} A_{ij} & \cdots & A_{ij} \\ & \cdots & \\ A_{ij} & \cdots & A_{ij} \end{pmatrix}$$

and when $i = j$,

$$P_{ij} = P_{ii} = \begin{pmatrix} Q_i & \cdots & Q_i \\ & \ddots & \vdots \\ J(Q_i) & & Q_i \end{pmatrix} .$$

We put

$$P(R) = \begin{pmatrix} P_{11} & \cdots & P_{1m} \\ & \cdots & \\ P_{m1} & \cdots & P_{mm} \end{pmatrix} .$$

Then $P = P(R)$ becomes a ring by usual matrix operations. Let p_{ij} be the element of P such that its (ij, ij) position is the unity of $P_{ij,ij}$ and all other positions are zero. Then $\{p_{11}, \ldots, p_{1n(1)}, \ldots, p_{m1}, \ldots, p_{mn(m)}\}$ is a complete set of orthogonal primitive idempotents of P; $P = p_{11}P \oplus \cdots \oplus p_{1n(1)}P \oplus \cdots \oplus p_{m1}P \oplus \cdots \oplus p_{mn(m)}P$.

We put

$$K_{ij} = \begin{pmatrix} K_{i1,j1} & \cdots & K_{i1,jn(j)} \\ & \cdots & \\ K_{in(i),j1} & \cdots & K_{in(i),jn(j)} \end{pmatrix}$$

where

$$K_{ik,jt} = \begin{cases} 0 & j \neq \sigma(i) \\ 0 & j = \sigma(i),\ t \leqq \rho(i) \\ S(P_{ik,jt}) & j = \sigma(i),\ t > \rho(i). \end{cases}$$

Namely

$$K_{ij} = \begin{pmatrix} 0 \cdots 0 \\ \cdots \\ 0 \cdots 0 \end{pmatrix} \ (j \neq \sigma(i))$$

$$K_{ij} = \begin{pmatrix} 0 \cdots 0 & S \cdots\cdot S \\ & \cdots & \\ 0 \cdots 0 & S \cdots\cdot S \end{pmatrix} \ (j = \sigma(i),\ S = S(P_{i1,\sigma(i)1})).$$

The following holds ([2]).

Theorem 1. *There is a ring epimorphism τ of P to R such that* $Ker\ \tau = \begin{pmatrix} K_{11} & \cdots & K_{1m} \\ & \cdots & \\ K_{m1} & \cdots & K_{mm} \end{pmatrix}$. *So, $R \simeq P/Ker\ \tau$.*

Remark. In

$$P_{i\sigma(i)} = \begin{pmatrix} P_{i1,\sigma(i)1} & \cdots & P_{i1,\sigma(i)n(\sigma(i))} \\ & \cdots & \\ P_{in(i),\sigma(i)1} & \cdots & P_{in(i),\sigma(i)n(\sigma(i))} \end{pmatrix}$$

we replace $P_{ij,\sigma(i)k}$ by $P^*_{ij,\sigma(i)k} = P_{ij,\sigma(i)k}/S(P_{ij,\sigma(i)k})$ for $1 \leq j \leq n(i)$, $\rho(i)+1 \leq k \leq n(\sigma(i))$, and denote it by $P^*_{i\sigma(i)}$.

In $P(R) = \begin{pmatrix} P_{11} & \cdots & P_{1m} \\ & \cdots & \\ P_{m1} & \cdots & P_{mm} \end{pmatrix}$

we replace $P_{i\sigma(i)}$ by $P^*_{i\sigma(i)}$ $(i = 1, \dots, m)$ and denote it by R^*. Then R^* canonically becomes a ring and isomorphic to R; so R^* is a representative matrix ring of R. We identify R with R^* or $R/Ker\ \tau$.

We can easily show the following by using injective pairs.

Proposition 1. *If R is a left H-ring of type $(*)$, then the ring*

$$\begin{pmatrix} Q_1 A_{12} \cdots\cdots A_{1m} \\ A_{21} Q_2 A_{23} \cdots A_{2m} \\ \cdots\cdots\cdots\cdots \\ A_{m1} \cdots A_{m,m-1} Q_m \end{pmatrix}$$

is a QF-ring. So, R is a suitable extension of this QF-ring.

Theorem 2. *If R is not of type $(*)$, then there are basic left H-rings $T_1, T_2, \dots, T_n$ and ring epimorphisms $\phi_1: T_1 \to T_2$, $\phi_2: T_2 \to T_3, \dots, \phi_n: T_n \to R$ such that T_1 is of type $(*)$ and each $Ker\ \phi_i$ is a simple ideal of T_i.*

Proof. We prove by induction on n. When $m = 1$, R is represented as

$$R = \left(\begin{array}{cccccc} Q & \cdots & Q & \bar{Q} & \cdots & \bar{Q} \\ J & \ddots & \vdots & \vdots & & \vdots \\ \vdots & & Q & \vdots & & \vdots \\ \vdots & & J & \bar{Q} & \cdots & \bar{Q} \\ \vdots & & \vdots & \bar{J} & \ddots & \vdots \\ J & \cdots & J & \bar{J} & \cdots\ \bar{J} & \bar{Q} \end{array}\right)$$

where $Q = e_{11}Re_{11}$, $J = J(Q)$ and $\bar{Q} = Q/S(Q)$. We put

$$T_1 = \left(\begin{array}{cccccc} Q & \cdots & Q & Q & \cdots & Q \\ J & \ddots & \vdots & \vdots & & \vdots \\ \vdots & & Q & \vdots & & \vdots \\ \vdots & & J & Q & \cdots & Q \\ \vdots & & \vdots & J & \ddots & \vdots \\ J & \cdots & J & J & \cdots\ J & Q \end{array}\right).$$

Then T_1 is a basic left H-ring of type $(*)$. We can easily see that

$$\begin{pmatrix} \ddots & & S(Q) & \\ & \ddots & & \\ & & \ddots & \\ & 0 & & \ddots \end{pmatrix}$$

is a two sided ideal of T_1. So, the following T_t canonically become rings :

$$T_2 = \begin{pmatrix} Q & \cdots & Q & \bar{Q} \\ & \ddots & & Q \\ & & \ddots & \vdots \\ J & & & Q \end{pmatrix}, \quad T_3 = \begin{pmatrix} Q & \cdots & Q & \bar{Q} \\ & \ddots & & \bar{Q} \\ & & \ddots & Q \\ & & & \vdots \\ J & & & Q \end{pmatrix}$$

$$T_4 = \begin{pmatrix} Q & \cdots & Q & \bar{Q} \\ & \ddots & & \bar{Q} \\ & & \ddots & \bar{Q} \\ & & & Q \\ & & & \vdots \\ J & & & Q \end{pmatrix}, \ldots, \quad T_s = \begin{pmatrix} Q & \cdots & Q & \bar{Q} \\ & \ddots & & \bar{Q} \\ & & \ddots & \bar{Q} \\ & & & \vdots \\ J & & & \bar{Q} \end{pmatrix}$$

$$s = n(1)+1$$

$$T_{s+1} = \begin{pmatrix} Q & \cdots & \bar{Q} & Q \\ & \ddots & Q & \bar{Q} \\ & & \ddots & \bar{Q} \\ & & & \vdots \\ J & & Q & \vdots \\ & & & \bar{Q} \end{pmatrix}, \ldots, \quad T_u = \begin{pmatrix} Q & \cdots & Q & \bar{Q} & \cdots & \bar{Q} & \bar{Q} \\ & \ddots & & & & & \bar{Q} \\ & & Q & & & & \bar{Q} \\ & & & \bar{Q} & & & \vdots \\ J & & & \vdots & \ddots & & \vdots \\ & & & \bar{J} & & \ddots & \vdots \\ & & & J & & & \bar{Q} \end{pmatrix}$$

$$u = n(1)(n(1)-j).$$

Moreover, we can see that these T_i are basic left H-rings and there are canonical ring epimorphisms: $T_1 \to T_2 \to T_3 \to \cdots \to T_n \to R$ whose kernels are simple ideals.

Now, we assume that the proposition is true on $m = k-1$, and let $m = k$. First, we consider the case $\{\sigma(1), \ldots, \sigma(n)\} \subsetneqq \{1, \ldots, n\}$. Then we can assume that there exists $j \in \{2, \ldots, k\}$ such that $\sigma(1) = \sigma(2) = \cdots = \sigma(j) \neq \sigma(t)$ for $j < t \leqq k$, and $\rho(1) < \rho(2) < \rho(\ell)$ for all $3 \leqq \ell \leqq j$ if $j \geqq 3$. We shall consider the following cases:

1) $\sigma(1) \neq 1, 2$

2) $\sigma(1) = 1$,

3) $\sigma(1) = 2$.

When the case 1), R can be represented as

$$R = \left(\begin{array}{c|c|c} * & \begin{array}{c} A \cdots A\,\bar{A} \cdots\cdots\cdots\cdots \bar{A} \\ \cdots \qquad \cdots\cdots\cdots\cdots \\ A \cdots A\,\bar{A} \cdots\cdots\cdots\cdots \bar{A} \end{array} & * \\ \hdashline * & \begin{array}{c} B \cdots\cdots\cdots\cdots B\,\bar{B} \cdots \bar{B} \\ \cdots\cdots\cdots\cdots \qquad \cdots \\ B \cdots\cdots\cdots\cdots B\,\bar{B} \cdots \bar{B} \end{array} & * \\ \hdashline * & * & * \end{array}\right)$$

where $A = e_{11}Re_{\sigma(1)}$, $B = e_{21}Re_{\sigma(1)}$, $\bar{A} = A/S(A)$ *and* $\bar{B} = B/S(B)$.

By replacing $\begin{pmatrix} \bar{A} \cdots \bar{A} \\ \cdots \\ \bar{A} \cdots \bar{A} \end{pmatrix}$ in

$$R = \left(\begin{array}{c|c|c} * & \begin{array}{c} A \cdots A\,\bar{A} \cdots\cdots\cdots\cdots \bar{A} \\ \cdots \qquad \cdots\cdots\cdots\cdots \\ A \cdots A\,\bar{A} \cdots\cdots\cdots\cdots \bar{A} \end{array} & * \\ \hdashline * & \begin{array}{c} B \cdots\cdots\cdots\cdots B\,\bar{B} \cdots \bar{B} \\ \cdots\cdots\cdots\cdots \qquad \cdots \\ B \cdots\cdots\cdots\cdots B\,\bar{B} \cdots \bar{B} \end{array} & * \\ \hdashline * & * & * \end{array}\right)$$

by $\begin{pmatrix} A \cdots A \\ \cdots \\ A \cdots A \end{pmatrix}$ and we denote it by T:

$$T = \left(\begin{array}{c|c|c} * & \begin{array}{c} A \cdots\cdots\cdots\cdots A\,\bar{A} \cdots \bar{A} \\ \cdots\cdots\cdots\cdots \qquad \cdots \\ A \cdots\cdots\cdots\cdots A\,\bar{A} \cdots \bar{A} \end{array} & * \\ \hdashline * & \begin{array}{c} B \cdots\cdots\cdots\cdots B\,\bar{B} \cdots \bar{B} \\ \cdots\cdots\cdots\cdots \qquad \cdots \\ B \cdots\cdots\cdots\cdots B\,\bar{B} \cdots \bar{B} \end{array} & * \\ \hdashline * & * & * \end{array}\right).$$

Let f_{ij} be the element of T such that (ij, ij) position is the unity of Q_i and all other positions are zero. Then $\{f_{11}, \ldots, f_{1n(1)}, f_{21}, \ldots, f_{2n(2)}, \ldots, f_{m1}, \ldots, f_{mn(m)}\}$ is a complete set of orthogonal primitive idempotents. We can see that T is a basic left artinian ring such that

a) $J(f_{ij}T_T)_T \simeq f_{i,j+1}T_T$ for $i = 1, \ldots, m$, $j = 1, \ldots, m(i)-1$

b) $f_{i1}T_T$ is injective for $i \neq 2$.
As T is basic and $S(f_{11}T_T)_T \simeq \cdots \simeq S(f_{1n(1)}T_T)_T \simeq S(f_{21}T_T)_T \simeq \cdots \simeq S(f_{2n(2)}T_T)_T$, we see that $J(f_{1n(1)}T_T)_T \supseteqq f_{21}T_T$. It is easy to see that both $J(f_{1n(1)}T_T)$ and $f_{21}T$ canonically become right R-modules (note that $\rho(1) < \rho(2) < \rho(\ell)$ for $3 \leqq \ell \leqq j$ if $j \geqq 3$). Since $J(f_{1n(1)}T)_R$ is indecomposable and $f_{21}T_R$ is injective, we have that $J(f_{1n(1)}T_T)_T \simeq f_{21}T_T$. Thus T is a basic left H-ring.

We see that

$$\left(\begin{array}{c:cc:c:c} 0 & 0 & S(A) & 0 & 0 \\ \hdashline 0 & & 0 & & 0 \\ \hdashline 0 & & 0 & & 0 \end{array}\right)$$

is a two sided ideal of T. So, as in the proof of the case $n = 1$, together with the induction hypothesis, we can obtain desired basic left H-rings and epimorphisms.

In view of the proof above, the same proof works for 2) and 3), and also for the case $\{\sigma(1), \ldots, \sigma(n)\} = \{1, \ldots, n\}$.

As an immediate corollary of the theorem above, we obtain

Corollary. *Left H-rings are (left and) right artinian rings.*

By Proposition 1 and Theorems 1, 2, we see that left H-rings can be constructed by suitable extensions of QF-rings and their factors.

References

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[2] K. Oshiro: On Harada rings II, Math. J. Okayama Univ. 31 (1989), 179−188.

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