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SELF-DUALITY FOR FINITE NORMALIZING EXTENSIONS OF SKEW FIELDS

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In this paper we shall prove the following

Theorem. *If the ring R is a finite normalizing extension of a skew field, then R has weakly symmetric self-duality.*

Before giving the proof we shall require some definitions, and we should like to provide some examples.

All our rings have an identity $\neq 0$, and all our ring extensions and modules are unitary. If ${}_TM_R$ and ${}_SN_R$ are bimodules, then $\mathrm{Hom}_R(M, N)$ is an S-T-bimodule in the natural way. A ring extension $R \geqq D$ is *finite normalizing* if there is a subset $\{b_1, \ldots, b_n\} \subset R$ such that $R = \sum_{i=1}^{n} b_i D$ with $b_i D = D b_i$ for all $i = 1, \ldots, n$. We assume that the reader is familiar with the theory of (Morita-)duality as, e.g., presented in Anderson, Fuller [1, §§ 23, 24]. A ring R has *self-duality* if there is an R-bimodule which defines a duality; and R has *weakly symmetric self-duality* (wssd) if there is an R-bimodule U which defines a duality such that ${}_R Re/Je \simeq {}_R\mathrm{Hom}_R(eR/eJ_R, U_R)$ (with $J = \mathrm{Ra}(R)$) for every primitive idempotent $e \in R$. For other terminology we would refer to [1].

Finite normalizing extensions of skew fields occur quite frequently. The most common examples are the finite dimensional algebras over fields. Other examples are

(1) finite incidence rings over skew fields (Haack [10]; they coincide with Mitchell's tic tac toe rings over skew fields [14, 10.8]), in particular rings with quivers that are trees (Fuller, Haack [8]), and thus the hereditary artinian rings of the finite representation types A_n, D_m, E_6, E_7 and E_8 ($n \geqq 1$, $m \geqq 4$; see Dlab, Ringel [7] for definitions). For this class of rings the existence of wssd has been shown in [10].

(2) ℓ-hereditary semidistributive rings which are indecomposable (see Caballero [5] and the references cited there); the existence of wssd is shown there.

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(3) split exact trivial extensions of semisimple rings (see Camillo, Fuller, Haack [6]). The existence of wssd has already been observed by W. Müller [17] and has been shown in [6], for instance. See also Question (1) at the end of the paper.

(4) crossed monoid rings over skew fields. Let D be a skew field, M a finite monoid with the neutral element e, and $R = \bigoplus_{X \in M} R_X$ a strongly M-graded ring with $R_e = D$ (that is R is a ring with $R_X R_Y = R_{XY}$ for all $x, y \in M$). If, for all $x \in M$, $R_X = D\tilde{x} = \tilde{x}D \neq 0$ for some $\tilde{x} \in R_X$, then R is a *crossed monoid ring of M over D*. Particular examples of such rings are the *twisted monoid rings of M over D* (that is $d\tilde{x} = \tilde{x}d$ for all $x \in M$ and $d \in D$), the *skew monoid rings of M over D* (that is $\tilde{x}\tilde{y} = (xy)^{\sim}$ for all $x, y \in M$), and the *semigroup ring of M over D* (that is $d\tilde{x} = \tilde{x}d$ and $\tilde{x}\tilde{y} = (xy)^{\sim}$ for all $x, y \in M$ and $d \in D$). If R is the semigroup ring of M over D, then the existence of wssd has been proved in a more general setting in Fuller, Haack [9].

In most of the cases the proofs given for the examples mentioned above are rather lengthy and depend on the special situation which has been assumed. Here we shall give a relatively short proof in a more general situation.

Proof of the Theorem. Let $R \geqq D$ be a finite normalizing extension with a skew field D, and let $\{b_1, \ldots, b_n\} \subset R$ such that $R = \sum_{i=1}^{n} b_i D$ with $b_i D = D b_i$ $(i = 1, \ldots, n)$. Then $\{b_1, \ldots, b_n\}$ contains a basis of the D-vectorspace R, hence we can assume without restriction that $R = \bigoplus_{i=1}^{n} b_i D$. We shall show that the injective cogenerator $U_R = \mathrm{Hom}_D(R_D, D_D)_R$ can be made into an R-bimodule which defines a wssd.

Proof of the existence of self-duality. For $i = 1, \ldots, n$ there are automorphisms $\mu_i\colon D \to D$, $d \to \tilde{d}$ with $db_i = b_i\tilde{d}$. Let $v_j\colon R_D \to D_D$, $\sum_i b_i d_i \to d_j$ be the j-th projection; we claim that $U = \bigoplus_j Dv_j$ with $v_j d = \mu_j(d) v_j$, in particular $Dv_j = v_j D$ $(d \in D,\ j = 1, \ldots, n)$: Evidently $U = \bigoplus_j Dv_j$, and for $d \in D$ and $j, k = 1, \ldots, n$ we calculate $(v_j d)(b_k) = v_j(db_k) = \delta_{jk}\mu_k(d) = (\mu_j(d) v_j)(b_k)$ (with the Kronecker symbol δ_{jk}), hence $v_j d = \mu_j(d) v_j$.

If $T = \mathrm{End}(U_R)$, then U is a T-R-bimodule, and from the adjointness of the Hom- and Tensor-functors we infer that $F\colon T \to \mathrm{Hom}_D(U_D, D_D)$, $\tilde{t} \to (u \to (\tilde{t}u)(1))$ is a D-T-biisomorphism with the inverse $G(t)(u)(r) = t(ur)$ $(t \in \mathrm{Hom}_D(U, D),\ u \in U,\ r \in R)$. Let $t_k\colon U_D \to D_D$, $\sum_j v_j d_j \to d_k$ be the k-th projection; then $\mathrm{Hom}_D(U_D, D_D) = \bigoplus_k Dt_k$ with $dt_k = t_k\mu_k(d)$, in particular $Dt_k = t_k D$ $(d \in D,\ k = 1, \ldots, n)$, as is shown with the same arguments as above. Denote $t'_k = G(t_k) \in T$; then $T = \oplus_k t'_k D$ with $dt'_k = t'_k \mu_k(d)$ $(d \in D, k = 1, \ldots, n)$.

There are $a_{ijk}, a'_{ijk}, a^*_{ijk}, a^{**}_{ijk} \in D$ $(i, j, k = 1, \ldots, n)$ such that $b_i b_j = \sum_\ell b_\ell a_{ij\ell}$, $v_k b_i = \sum_\ell v_\ell a'_{ki\ell}$, $t'_j v_k = \sum_\ell v_\ell a^*_{jk\ell}$, $t'_i t'_j = \sum_\ell t'_\ell a^{**}_{ij\ell}$. In order to prove that R has self-duality it is now sufficient to show that $a_{ijk} = a^{**}_{ijk}$ $(i, j, k = 1, \ldots, n)$. Then H: $T \to R$, $\sum_i t'_i d_i \to \sum_i b_i d_i$ is an isomorphism of rings and our statement follows because R is artinian and U_R is of finite length (see, e.g., Azumaya [4, Theorem 6], Morita [15, Theorem 6.3], or B. Müller [16, Theorem 7]).

For $i, j, k = 1, \ldots, n$ we evaluate as follows.

$(v_k b_i)(b_j) = (\sum_\ell v_\ell a'_{ki\ell})(b_j) = \mu_j(a'_{kij})$ and
$(v_k b_i)(b_j) = v_k(b_i b_j) = v_k(\sum_\ell b_\ell a_{ij\ell}) = a_{ijk}$;

$(t'_j v_k)(b_i) = (t'_j \cdot v_k b_i)(1) = F(t'_j)(v_k b_i) = t_j(\sum_\ell v_\ell a'_{ki\ell}) = a'_{kij}$ and
$(t'_j v_k)(b_i) = (\sum_\ell v_\ell a^*_{jk\ell})(b_i) = \mu_i(a^*_{jki})$;

$(t'_i t'_j \cdot v_k)(1) = (t'_i \cdot t'_j v_k)(1) = F(t'_i)(t'_j v_k) = t_i(\sum_\ell v_\ell a^*_{jk\ell}) = a^*_{jki}$ and
$(t'_i t'_j \cdot v_k)(1) = F(\sum_\ell t'_\ell a^{**}_{ij\ell})(v_k) = \sum_\ell t_\ell(v_k)\mu_k^{-1}(a^{**}_{ij\ell}) = \mu_k^{-1}(a^{**}_{ijk})$.
Hence $a_{ijk} = \mu_j \mu_i \mu_k^{-1}(a^{**}_{ijk})$ $(i, j, k = 1, \ldots, n)$.

On the other hand there holds that $db_i b_j = b_i b_j \mu_j \mu_i(d) = \sum_k b_k a_{ijk} \mu_j \mu_i(d)$ and $db_i b_j = d \cdot \sum_k b_k a_{ijk} = \sum_k b_k \mu_k(d) a_{ijk}$, thus, comparing coefficients, $\mu_k(d) a_{ijk} = a_{ijk} \mu_j \mu_i(d)$ $(d \in D,\ i, j, k = 1, \ldots, n)$. If we put $d = \mu_k^{-1}(a_{ijk})$, then $a^2_{ijk} = a_{ijk} \mu_j \mu_i \mu_k^{-1}(a_{ijk})$, hence $a_{ijk} = \mu_j \mu_i \mu_k^{-1}(a_{ijk})$ for $i, j, k = 1, \ldots, n$ (we can cancel if $a_{ijk} \neq 0$).

Then $a_{ijk} = a^{**}_{ijk}$ $(i, j, k = 1, \ldots, n)$, thus R has self-duality.

Proof of the existence of wssd. Via H the T-R-bimodule U is an R-bimodule which defines a self-duality. Denote $J = \mathrm{Ra}(R)$ and $V = \mathrm{So}(U)$. Note that V is an R/J-bimodule because V is the left and the right annihilator of J. We have shown in [11, II.5] that U defines a wssd if and only if $\tilde{e} V \tilde{e} \neq 0$ for every central primitive idempotent $\tilde{e} \in R/J$. (From this wssd follows rather immediately if R is a finite dimensional algebra over D, since in this case $(ru)(r') = u(r'r)$ $(r, r' \in R,\ u \in U)$, as is easily shown.)

It is no problem to show that $R/J \geqq (D+J)/J$ is a finite normalizing extension. We shall identify $(D+J)/J \simeq D/(D \cap J) = D/0$ with D, and we shall denote residues modulo J by $\bar{\ }$. From $\bar{R} = \sum_{i=1}^n \bar{b}_i D$ it follows that there is a subset $K \subset \{1, \ldots, n\}$ such that $\bar{R} = \oplus_K \bar{d}_k D$ with $d\bar{b}_k = \bar{b}_k \mu_k(d)$ $(d \in D,\ k \in K)$.

It is well-known that Φ: $V \to \mathrm{Hom}_D(\bar{R}_D, D_D)$, $u \to (\bar{r} \to u(r))$ is a D-R-biisomorphism. If w_k : $\bar{R}_D \to D_D$, $\sum_{i \in K} \bar{b}_i d_i \to d_k$ $(k \in K)$, then $\mathrm{Hom}_D(\bar{R}_D, D_D)$

$= \bigoplus_K Dw_k$ with $w_k d = \mu_k(d) w_k$ $(d \in D,\ k \in K)$. Let $\tilde{v}_k = \Phi^{-1}(w_k) \in V$; then $V = \bigoplus_K D\tilde{v}_k$ with $\tilde{v}_k d = \mu_k(d)\tilde{v}_k$ $(d \in D,\ k \in K)$. There are $d'_{kj} \in D$ with $\tilde{v}_k = \sum_{j=1}^{n} v_j d'_{kj}$ $(k \in K)$, and we calculate for $k \in K,\ i, j = 1,\dots, n$:

$$\tilde{v}_k(b_i) = (\textstyle\sum_{j=1}^{n} v_j d'_{kj})(b_i) = \mu_i(d'_{ki}),\ t_j(\tilde{v}_k) = t_j(\sum_{i=1}^{n} v_i d'_{ki}) = d'_{kj}. \quad (1)$$

Let $\tilde{e} = \sum_{j \in K} \overline{b_j} d'_j \in \overline{R}$ be a central primitive idempotent. Then $e = \sum_{j \in K} b_j d'_j \in R$ with $\overline{e} = \tilde{e}$. From $\tilde{e}$ central we infer $\sum_{j \in K} \overline{b_j} \mu_j(d) d'_j = \sum_{j \in K} d\overline{b_j} d'_j = d\tilde{e} = \tilde{e}d = \sum_{j \in K} \overline{b_j} b'_j d$ $(d \in D)$, hence, comparing coefficients,

$$d'_j d = \mu_j(d) d'_j \quad (d \in D,\ j \in K); \qquad (2)$$

in particular $d'^2_j = \mu_j(d'_j) d'_j$, thus

$$d'_j = \mu_j(d'_j) \quad (j \in K). \qquad (3)$$

Now we shall show that $\tilde{e}v = v\tilde{e}$ for $v \in V$. It is sufficient to show that $\tilde{e}\tilde{v}_k = \tilde{v}_k\tilde{e}$ $(k \in K)$ because $d\tilde{e} = \tilde{e}d$ $(d \in D)$ and $V = \bigoplus_K D\tilde{v}_k$ with $D\tilde{v}_k = \tilde{v}_k D$ $(k \in K)$. In order to prove this we have to show that $(\tilde{e}\tilde{v}_k)(b_i) = (\tilde{v}_k\tilde{e})(b_i)$ $(k \in K,\ i = 1,\dots, n)$. Let $\tilde{v}_k b_i = \sum_{\ell \in K} \tilde{v}_\ell \tilde{a}_{ki\ell}$ with $\tilde{a}_{ki\ell} \in D$ $(k \in K,\ i = 1,\dots, n)$. If we use the identification of R and T via H and that V is the annihilator of J, then we calculate

$$\begin{aligned}(\tilde{e}\tilde{v}_k)(b_i) = (e\tilde{v}_k b_i)(1) = F(e)(\tilde{v}_k b_i) = (\textstyle\sum_{j \in K} t_j d'_j)(\sum_{\ell \in K} \tilde{v}_\ell \tilde{a}_{ki\ell}) &= \\ \textstyle\sum_{j,\ell \in K} \mu_j^{-1}(d'_j) \cdot t_j(\tilde{v}_\ell) \cdot \tilde{a}_{ki\ell} &= \quad (1) \\ \textstyle\sum_{j,\ell \in K} \mu_j^{-1}(d'_j) d'_{\ell j} \tilde{a}_{ki\ell} &= \quad (3) \\ \textstyle\sum_{j,\ell \in K} d'_j d'_{\ell j} \tilde{a}_{ki\ell} &\end{aligned}$$

and if we use in addition that $er - re \in J$ $(r \in R)$, then we calculate

$$\begin{aligned}(\tilde{v}_k\tilde{e})(b_i) = (\tilde{v}_k e)(b_i) = \tilde{v}_k(eb_i) = \tilde{v}_k(b_i e) = (\tilde{v}_k b_i)(e) &= \\ (\textstyle\sum_{\ell \in K} \tilde{v}_\ell \tilde{a}_{ki\ell})(e) = \sum_{\ell \in K} \tilde{v}_\ell(e\tilde{a}_{ki\ell}) &= \\ \textstyle\sum_{\ell \in K} \tilde{v}_\ell(\sum_{j \in K} b_j d'_j)\tilde{a}_{ki\ell} &= \quad (1) \\ \textstyle\sum_{j,\ell \in K} \mu_j(d'_{\ell j}) d'_j \tilde{a}_{ki\ell} &= \quad (2) \\ \textstyle\sum_{j,\ell \in K} d'_j d'_{\ell j} \tilde{a}_{ki\ell} &\end{aligned}$$

$(k \in K,\ i = 1,\dots, n)$. Hence $\tilde{e}v = v\tilde{e}$ for $v \in V$.

From $\tilde{e}V \neq 0$ (because V is a faithful R/J-module) we infer now the existence of a wssd, and the theorem is proved. □

Corollary. *Every factor ring of R has wssd.*

Proof. If $A < R$ is a two-sided ideal, then it is no problem to show that $R/A \geqq (D+A)/A \simeq D/(D \cap A) = D/0 \simeq D$ is a finite normalizing extension of a skew field, thus our statement follows from the Theorem. □

The following two questions arise.

(1) Let $R \geqq D$ be a finite normalizing extension of the skew field D. Does R have *almost symmetric self-duality* (assd), i.e. is there an R-bimodule U which defines a self-duality such that $\mathrm{So}(U) \simeq R/\mathrm{Ra}(R)$ as R-bimodules? (Equivalently: Is there some $u \in \mathrm{So}(U)$ with $\mathrm{So}(U) = Ru = uR$ and $ru = ur$ for all $r \in R$? Note that from wssd there follows the existence of some $u \in \mathrm{So}(U)$ with $\mathrm{So}(U) = Ru = uR$; and compare also [16, Section 2].) We have introduced this concept, which is derived from Azumaya [3], in our paper [12]. It is stronger than wssd (i.e. if U defines an assd, then U defines a wssd), and it plays an important role in the representation theory of rings, as one sees from the following argument. Let ${}_SV_S$ define an assd and denote $J = \mathrm{Ra}(S)$, $B = S/J$. Then $\mathrm{Hom}_S(J^i/J^{i+1}{}_S, B_S) \simeq \mathrm{Hom}_S({}_SJ^i/J^{i+1}, {}_SB)$ as S-bimodules for $i \geqq 0$, as can be derived from Rosenberg, Zelinsky [18] (see [12, Corollary 9(2)]). In the case of $i = 1$ this biisomorphism is, or can be transformed into, the duality condition used by W. Müller [17], Dlab, Ringel [7], Auslander, Platzeck, Reiten [2] and others. In certain cases this biisomorphism is equivalent to the existence of assd (see [2, 5.7]); we have generalized this in our paper [13].

It can be shown that finite dimensional algebras over fields have assd (the proof uses that semisimple algebras are symmetric Frobenius algebras). From [6, 3.1, 3.2] there follows that the rings of Example (3) have assd. Hereditary artinian rings of the finite representation types A_n, D_m, E_6, E_7 and E_8 ($n \geqq 1$, $m \geqq 4$; cf. Example (1)) also have assd (see [13, 4.7(2)]; the proof needs different methods than the ones which we have used here). It can be shown that hereditary artinian rings of the finite representation types B_m, C_m ($m \geqq 2$) and F_4 have wssd, but in general they do not have assd. In the case of $m = 2$ an example has been given in [13, 5.9(3)], which can easily be generalized to the cases of $m \geqq 3$ and F_4.

(2) Is it possible to generalize the Theorem? What happens, e.g., if we substitute the skew field by a finite product of complete noetherian valuation rings, called $\tilde{D}$? (See below for definitions. Note that $\tilde{D}$ has wssd) In this case there are the following difficulties which must be solved.

(i) In general $\tilde{D}$ has zero divisors, so we cannot cancel, as we have done in the proofs of $a_{ijk} = \mu_j\mu_i\mu_k^{-1}(a_{ijk})$ and $d_j' = \mu_j(d_j')$.

(ii) In general $R_{\tilde{D}}$ is not a free module, so we cannot assume that the μ_i are automorphisms of $\tilde{D}$.

(iii) If $R = \sum_{i=1}^{n} b_i\tilde{D}$ with $b_i\tilde{D} = \tilde{D}b_i$ (i = 1,..., n), then in general there is not a subset $I \subset \{1,..., n\}$ such that $R = \bigoplus_I b_i\tilde{D}$.

So, in order to derive results, it is necessary to make additional assumptions and to modify the calculations.

The most common examples of such rings are Artin algebras (see [11, II. 7]) and complete noetherian serial rings, in particular artinian serial rings.

(Definitions. A module is called *uniserial* if its submodules are linearly ordered with respect to inclusion. Let S be a ring and denote $J = \mathrm{Ra}(S)$. A module M_S is called *complete* if $M \simeq \varprojlim(M/MJ^i)$, the inverse limit (see Rotman [19, Chapter 2] for details), and S is a *complete noetherian valuation ring* if S_S and $_SS$ are complete, noetherian and uniserial. The ring S is called *serial* if S_S and $_SS$ are finite direct sums of uniserial modules; and S is an *Artin algebra* if S is artinian and a finitely generated module over its centre.)

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