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SOME CHARACTERIZATIONS OF π-REGULAR RINGS OF BOUNDED INDEX

Dedicated to Professor Manabu Harada on his 60th birthday

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In this paper, all rings contain an identity element. A ring R is said to be *of bounded index* (*of nilpotence*) if there is a positive integer n such that $a^n = 0$ for all nilpotent elements a of R. The least such integer is called the *index* of R, and we denote it by $i(R)$. Recall that R is said to be *π-regular* if for each element a of R, there exists a positive integer n and an element x of R such that $a^n = a^n x a^n$. The purpose of this paper is to give some characterizations of π-regular rings of bounded index. We show that a ring R is a π-regular ring of index at most n if and only if the endomorphism ring $End_R(M)$ of any cyclic module M is of index at most n. From this result, we obtain that every finite extension of a π-regular ring of bounded index is also a π-regular ring of bounded index. We also show that a ring R of bounded index is π-regular if and only if every prime factor ring of R is Artinian. Using this result, we prove that if S is a π-regular ring of bounded index and if S is a finite normalizing extension of a ring R, then R is also a π-regular ring of bounded index.

Let R be a ring of index n. Then it is proved in [2] that the following statements are equivalent :

1) R is a π-regular ring.
2) It holds that $a^nR = a^{n+1}R$ for all $a \in R$.
3) It holds that $Ra^n = Ra^{n+1}$ for all $a \in R$.

Thus, if R is a π-regular ring of index n, then every factor ring of R is of index at most n. However the converse is not true. For example, for any positive integer k, the k-th Weyl algebra $A_k(Q)$ over the field Q of rational numbers is a simple domain, but it is not π-regular. Nevertheless, we can show that the center of such a ring is π-regular.

Proposition 1. *If every factor ring of a ring R is of index at most n, then the center $Z(R)$ of R is a π-regular ring of index at most n.*

Proof. Let a be an element of $Z(R)$. Consider the ideal $I = a^{n+1}R$. Then $a+I$ is a nilpotent element of the ring R/I. Since R/I is of index at most n by hypothesis, $a^n \in I$. Then there exists an element $b \in R$ such that $a^n = a^{2n}b$. Hence a^n is strongly regular, and so by [2, Lemma 1] there exists $z \in Z(R)$ such that $a^n = a^{2n}z$. This proves that $Z(R)$ is a π-regular ring of bounded index.

Corollary 1. *Let R be a PI-ring. If every factor ring of R is of index at most n, then R is a π-regular ring of index at most n.*

Proof. By [5, Theorem 2.3] it suffices to prove that every prime factor ring of R is a simple Artinian ring. So, without loss of generality, we may assume that R is a prime ring. By Proposition 1 the center $Z(R)$ of R is π-regular, and so $Z(R)$ is a field. Hence R is a simple Artinian ring by [10, Corollary to Theorem 2].

To characterize a π-regular ring of bounded index in this direction, we need consider the endomorphism rings of cyclic modules.

Theorem 1. *The following statements are equivalent:*

1) *R is a π-regular ring of index at most n.*

2) *For any cyclic module M, $i(End_R(M)) \leqq n$.*

In this case, for any module N generated by m elements, it holds that $i(End_R(N)) \leqq i(M_m(R))$.

Proof. Suppose that 1) holds. Let M be a cyclic right R-module. Then M is isomorphic to R/K for some right ideal K of R. Let $I_R(K)$ denotes the idealizer of K in R. Then $End_R(M)$ is isomorphic to the ring $I_R(K)/K$. We claim that $I_R(K)/K$ is of index at most n. Let a be an element of $I_R(K)$ with $a^p \in K$ for some positive integer p. Since R is a π-regular ring of index at most n, there exists $x \in R$ such that $a^n = a^{n+1}x$. If $p > n$, then $a^{p-1} = a^p x \in K$, because K is a right ideal of R. Continuing this process, we obtain that $a^n \in K$.

Let N be a right R-module generated by $a_1, a_2, \dots, a_m$ and let $K = \{z \in M_m(R) \mid (a_1, \dots, a_m)z = (0, \dots, 0)\}$. Given $g \in End_R(N)$, we can write $g(a_1, \dots, a_m) = (a_1, \dots, a_m)(r_{ij})$ for some $(r_{ij}) \in I_{M_m(R)}(K)$. Then the map ψ: $End_R(N) \to I_{M_m(R)}(K)/K$ defined by $\psi(g) = (r_{ij}) + K$ is a ring isomorphism. By [11, Theorem 5] $M_m(R)$ is also a π-regular ring of bounded index. Hence, by the same way as above, we can prove that the index of $I_{M_m(R)}(K)/K$ is less than or equal to $i(M_m(R))$.

2) $\Rightarrow$ 1). Let a be an element of R. Consider the cyclic module

$M = R/a^{n+1}R$. Then $End_R(M)$ is isomorphic to $I_R(a^{n+1}R)/a^{n+1}R$. Since $a \in I_R(a^{n+1}R)$ and $a^{n+1} \in a^{n+1}R$, we conclude that $a^n \in a^{n+1}R$. This implies that R is a π-regular ring of index at most n.

A ring R is called a *strongly regular ring* if R is a von Neumann regular ring and R has no non-zero nilpotent element. As an immediate corollary of Theorem 1, we have

Corollary 2. *The following statements are equivalent :*

1) *R is a strongly regular ring.*

2) *For any cyclic module M, $End_R(M)$ has no non-zero nilpotent elements.*

In this case, for any module N generated by m elements, we have $i(End_R(N)) \leqq m$.

Let R be a subring of a ring S. If S is finitely generated as a right R-module, S is called a *finite extension* of R.

Corollary 3. *Let S be a finite extension of a ring R. If R is a π-regular ring of bounded index, then so is S.*

Proof. Let M be a cyclic right S-module. Let S be generated by m elements as a right R-module. Then M is generated by m elements as a right R-module. By Theorem 1, $i(End_R(M)) \leqq i(M_m(R))$, and by [11, Theorem 5] $n = i(M_m(R))$ is finite. Since $End_S(M)$ is a subring of $End_R(M)$, we obtain that $i(End_S(M)) \leqq n$. Again, by Theorem 1, S is a π-regular ring of index $i(S) \leqq n$.

We shall sharpen [7, Theorem 2.3].

Proposition 2. *Let R be a π-regular ring of index n and let P be a prime ideal of R. Then R/P is isomorphic to a matrix ring $M_k(D)$ for some division ring D and some $k \leqq n$.*

Proof. Since $S = R/P$ is π-regular, the Jacobson radical J of S is a nil ideal. Hence $x^n = 0$ for all $x \in J$. Since S is a prime ring, $J = 0$ by [6, Lemma 1.1]. Moreover, by [8, Lemma 2], S has no infinite set of orthogonal idempotents. By [7, Theorem 2.1] S is a simple Artinian ring (of index at most n), and hence S is isomorphic to a matrix ring $M_k(D)$ for some division ring D and some $k \leqq n$.

Combining this proposition with [5, Theorem 2.1], we obtain

Theorem 2. *Let R be a ring of bounded index. Then the following statements are equivalent:*

1) *R is π-regular.*
2) *All prime factor rings of R are Artinian.*

According to [5, Theorem 2.3], a PI-ring R is π-regular if and only if all prime factor rings of R are Artinian. We shall show that a similar result of Corollary 3 holds for rings all of whose prime factor rings are Artinian. We say R is *strongly π-regular* if for each $a \in R$ there exists a positive integer n such that $a^nR = a^{n+1}R$. By [4, Théorème 1] this definition is left-right symmetric, and so such a ring is π-regular. The following is an easy consequence of [1, Theorem 1.1].

Proposition 3. *The following statements are equivalent:*

1) *For each $n \geqq 1$, $M_n(R)$ is strongly π-regular.*
2) *Every finite extension of R is strongly π-regular.*

Let R be a ring whose prime factor rings are Artinian. By virtue of [5, Theorem 2.1], R satisfies the condition 1) of Proposition 3. Hence we have

Corollary 4. *Let R be a ring whose prime factor rings are Artinian. Then every finite extension of R is strongly π-regular.*

Let S be a finite extension of a ring R. If all prime factor rings of S are Artinian, is R strongly π-regular ? We shall show that this is true when S is a finite normalizing extension of R. Recall that S is called a *finite normalizing extension* of R if there exists finitely many elements $x_1, x_2, \ldots, x_n$ in S such that $S = Rx_1 + Rx_2 + \cdots + Rx_n$ and $Rx_i = x_iR$ for all $i = 1, 2, \ldots, n$.

Proposition 4. *Let S be a finite normalizing extension of a ring R. If S is a ring whose prime factor rings are Artinian, then so is R.*

Proof. Let P be a prime ideal of R. By [3, Theorem 2.3] there is a prime ideal Q of S such that P is one of the minimal primes over $Q \cap R$. By hypothesis, S/Q is a simple Artinian ring. Since S/Q is a finite normalizing extension of $R/(Q \cap R)$, $R/(Q \cap R)$ is right Artinian by [9, Proposition 5 (iii)]. Since $Q \cap R$ is semiprime, $R/(Q \cap R)$ is a semisimple

Artinian ring. Then R/P is isomorphic to one of the simple components of $R/(Q \cap R)$. This proves that R/P is a simple Artinian ring.

By virtue of Theorem 2, we have

Corollary 5. *Let S be a finite normalizing extension of a ring R. If S is a π-regular ring of bounded index, then so is R.*

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