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ON EXTENSIONS OF RINGS WITH FINITE ADDITIVE INDEX

To the memory of Professor Shigeaki Tôgô

YASUYUKI HIRANO

In [1] we proved that if the additive group of the center Z of a ring R has a finite group-theoretic index in the additive group of R, then R has an ideal I contained in Z such that R/I is a finite ring. The purpose of this paper is to extend this result for extensions of rings with finite additive index. As an application of it, we prove that if a derivation d of an infinite simple ring has only finitely many values, then $d = 0$.

For a ring R, R^+ denotes the additive group of R. We shall prove the main theorem of this paper.

Theorem 1. *Let R be a subring of a ring S. Suppose that R^+ has a finite index in S^+. Then there exists an ideal I of S contained in R such that S/I is a finite ring.*

Proof. Consider the homomorphism $g: R \to End(S^+/R^+)$ defined by $g(r)(s+R^+) = rs+R^+$ for all $r \in R$ and $s+R^+ \in S^+/R^+$. Since S^+/R^+ is a finite group, $End(S^+/R^+)$ is a finite ring. Hence $Ker(g) = \{r \in R \mid rS \subseteq R\}$ has a finite index in R^+. Similarly, $\{r \in R \mid Sr \subseteq R\}$ has a finite index in R^+. Hence $I = \{r \in R \mid Sr \subseteq R$ and $rS \subseteq R\}$ has a finite index in R^+. Let n be the index of R^+ in S^+ and let $S^+/R^+ = \{a_1+R^+, a_2+R^+, \dots, a_n+R^+\}$. For each i, consider the map $f_i: I \to \mathrm{End}(S^+/R^+)$ defined by $f_i(r)(s+R^+) = a_irs+R^+$ for all $r \in I$ and $s+R^+ \in S^+/R^+$. Then each f_i is an additive map, and so the additive subgroup $Ker(f_i)$ has a finite index in I. Hence $I' = \bigcap_{i=1}^{n} Ker(f_i)$ has a finite index in R^+. Let r be an arbitrary element of I'. Then $rS \subseteq R$ and $a_irS \subseteq R$ for all $i = 1, 2, \dots, n$, and so $SrS \subseteq R$. Now it is easy to see that $I' = \{r \in R \mid SrS \subseteq R\} \cap I$. Therefore the ideal $J = I'+SI'+I'S+SI'S$ of S is contained in R, and S/J is a finite ring.

Corollary 1. *Let R be a subring of an infinite simple ring S. If R^+ has a finite index in S^+, then $S = R$.*

Corollary 2. *Let R be an infinite simple ring with identity e. If S is an extension of R and if R^+ has a finite index in S^+, then S is the direct sum of R and a finite ring.*

Proof. By Theorem 1, S has an ideal I contained in R such that S/I is a finite ring. Since R is an infinite simple ring, I must coinside with R. Thus R is an ideal of S, and so e is a central idempotent of S. Now our assertion is clear.

Corollary 3. *Let S be a ring which has no non-zero finite homomorphic images, and let d be a derivation of S. If d has only finitely many values in S, then $d = 0$.*

Proof. Let $Im(d) = \{s_1, s_2, \ldots, s_n\}$. For each $i = 1, 2, \ldots, n$, take an element $a_i \in S$ such that $d(a_i) = s_i$. Since d is a derivation of S, $R = \{a \in S \mid d(a) = 0\}$ is a subring of S. Now we can easily see that $S^+/R^+ = \{a_1+R^+, a_2+R^+, \ldots, a_n+R^+\}$. Therefore, by Theorem 1, S has an ideal I contained in R such that S/I is a finite ring. Then, by hypothesis, we conclude that $S = R$.

As an immediate consequence of Corollary 3, we have

Corollary 4. *Let S be a ring which has no non-zero finite homomorphic images, and let d denote the inner derivation of S induced by an element x of S. If $Im(d)$ is a finite subset of S, then x is contained in the center of S.*

Remark. In Corollary 3, d cannot be replaced by an additive map of S, and hence, in Theorem 1, R cannot be replaced by an additive subgroup with finite index. For example, let $K = GF(p)$ where p is a prime number, and let $K(x)$ be the field of rational functions in one variable over K. Then there exists a K-subspace L of $K(x)$ such that $K(x) = K \oplus L$. The projection $p: K(x) \to K$ defined by this decomposition is a non-zero additive map and $Im(p)\ (= K)$ is a finite subset of $K(x)$.

References

[1] Y. Hirano, On a problem of Szász, Bull. Austral. Math. Soc. 40 (1989), 363–364.

DEPARTMENT OF MATHEMATICS
OKAYAMA UNIVERSITY
OKAYAMA 700, JAPAN

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