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ON NEAR-RINGS WITH DERIVATION

Dedicated to Professor Takasi Nagahara on his 60th birthday

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Throughout, N will represent a zero-symmetric left near-ring, and A a non-zero ideal of N. Let $d: x \to x'$ be a derivation of N, i.e., an endomorphism of $(N, +)$ satisfying the "product rule" $(xy)' = xy' + x'y$ for all $x, y \in N$. An element x of N with $x' = 0$ will be called a constant. As usual, for $x, y \in N$, we write $[x, y] = xy - yx$, $x \circ y = xy + yx$ and $(x, y) = x + y - x - y$. The derivation d will be called *commuting* (resp. *semi-commuting*) on A if $[a, a'] = 0$ (resp. $[a, a'] = 0$ or $a \circ a' = 0$) for all $a \in A$. Given a subset S of N, we put $V_N(S) = \{x \in N \mid xs = sx \text{ for all } s \in S\}$. Finally, N will be called *prime* if $x, y \in N$ and $xNy = 0$ imply that $x = 0$ or $y = 0$. As for terminologies used here without mension, we refer to G. Pilz [3].

We consider the following conditions :

1) $A' \subseteq V_N(A')$, namely $[a', b'] = 0$ for all $a, b \in A$.
2) $a - a' \in V_N(A)$ for all $a \in A$, and $A' \subseteq A$.
3) $a + a' \in V_N(A)$ for all $a \in A$, and $A' \subseteq A$.

Our present objective is to prove the following theorems.

Theorem 1. *Suppose that A contains no non-zero zero-divisors of N. If $0 \neq A' \subseteq A$ and d is semi-commuting on A, then $(N, +)$ is abelian.*

Theorem 2. *Let N be a prime near-ring. If $A' \neq 0$, then the condition* 1) *implies that $(N, +)$ is abelian. If, furthermore, N is 2-torsion-free, then the condition* 1) *implies that N is a commutative ring.*

Theorem 3 *Let N be a prime near-ring. Then each of the conditions* 2), 3) *implies that $(N, +)$ is abelian.*

Obviously, Theorem 1 includes [1, Theorem 1], Theorem 2 generalizes [1, Theorems 2 and 3], and Theorem 3 is a partial extension of [1, Theorem 4].

In preparation for proving our theorems, we state four lemmas.

Lemma 1. *Let $a, b, c \in N$. If a and $a+a \in V_N(\{b, c, b+c\})$, then $a(b,c) = 0$.*

Proof. In fact, $ab+ab+ac+ac = ba+ba+ca+ca = b(a+a)+c(a+a) = (a+a)b+(a+a)c = (a+a)(b+c) = (b+c)(a+a) = (b+c)a+(b+c)a = a(b+c)+a(b+c) = ab+ac+ab+ac$.

Lemma 2. *Let u be an element of N which is not a left zero-divisor. If either $[u,u'] = 0$ or $u \circ u' = 0$, then $(u,x)' = 0$ for all $x \in N$.*

Proof. In view of [1, Lemma 2], it remains only to prove the case that $u \circ u' = 0$. Let $x \in N$. Then $uu'+ux'+u'u+u'x = (u(u+x))' = (u^2+ux)' = uu'+u'u+ux'+u'x$. This together with $uu'+u'u = 0$ implies that $u(u,x)' = u(u'+x'-u'-x') = 0$, and therefore $(u,x)' = 0$.

Lemma 3. *Let N be a prime near-ring.*

(1) *If $(A, +)$ is abelian, then so is $(N, +)$.*

(2) *If $(A, \cdot)$ is commutative, then so is $(N, \cdot)$.*

Proof. (1) Since $(A, +)$ is abelian, it is easy to see that $A(x,y) = 0$ for all $x, y \in N$. Then, noting that N is prime, we get $(x,y) = 0$.

(2) Let $a, b \in A$, and $x, y \in N$. Since $(A, \cdot)$ is commutative, $ab[x,y] = abxy - abyx = baxy - byax = axby - axby = 0$, so that $A^2[x,y] = 0$. Then N being prime, we get $[x,y] = 0$.

Lemma 4. *Let N be a prime near-ring, and $x \in N$.*

(1) *Let $a \in A$, and z a non-zero element of $V_N(A)$. If $za = 0$ (resp. $az = 0$), then $a = 0$.*

(2) *If $V_N(A)$ contains a non-zero element z such that $z+z \in V_N(A)$ then $(N, +)$ is abelian.*

(3) *If z is a non-zero element of $V_N(A)$ and $zx \in V_N(A)$ (resp. $xz \in V_N(A)$), then x is in $V_N(A)$.*

(4) *Let $A' \neq 0$. If $xA' = 0$ (resp. $A'x = 0$), then $x = 0$.*

(5) *Let $(N, +)$ be 2-torsion-free. If $A' \neq 0$, then $A'' \neq 0$.*

Proof. (1) This is clear by $zAa = 0$ (resp. $aAz = 0$).

(2) Let $a, b \in A$. Since z and $z+z \in V_N(A)$, we get $z(a,b) = 0$ (Lemma 1). Hence $(a,b) = 0$, by (1). This proves that $(A, +)$ is abelian, so that $(N, +)$ is abelian, by Lemma 3 (1).

(3) If $zx \in V_N(A)$ (resp. $xz \in V_N(A)$), then $z[x,b] = [zx,b] = 0$

(resp. $z[x, b] = [xz, b] = 0$) for all $b \in A$. Since $[x, b] \in A$, (1) shows that $[x, b] = 0$, and so $x \in V_N(A)$.

(4) Choose an element a of A with $a' \neq 0$. If $xA' = 0$ (resp. $A'x = 0$), then $xAa' = x(Aa)' = 0$ (resp. $a'Ax = (aA)'x = 0$ by [1, Lemma 1]), and so $x = 0$.

(5) Suppose, to the contrary, that $A'' = 0$. Then, for each a, $b \in A$, $0 = (ab)'' = 2a'b'$, whence $a'b' = 0$ follows. Hence $a'A' = 0$, and we see that $a' = 0$ by (4). But this is a contradiction.

We are now ready to complete the proofs of our theorems.

Proof of Theorem 1. Let $a \in A$, and $x \in N$. Then $(a, x)' = 0$ by Lemma 2. Further, for any $b \in A$ with $b' \neq 0$, $b'(a, x) = (b(a, x))' = (ba, bx)' = 0$, and therefore $(a, x) = 0$. The rest of the proof is clear by Lemma 3 (1).

Proof of Theorem 2. Let a, b, $c \in A$. Since a' and $a'+a' \in V_N(A)$, we get $a'(b, c)' = 0$ (Lemma 1). Hence $A'(b, c)' = 0$, and so $(b, c)' = 0$ by Lemma 4 (4). Consequently, $a'(b, c) = (a(b, c))' = (ab, ac)' = 0$. Again by Lemma 4 (4), we obtain $(b, c) = 0$. This shows that $(A, +)$ is abelian, and so $(N, +)$ is abelian by Lemma 3 (1).

Henceforth, we assume further that N is 2-torsion-free. Then, in view of Lemma 3 (2), it suffices to show that $(A, \cdot)$ is commutative. By Lemma 4 (5), A contains an element a with $a'' \neq 0$. Let b, c, x be arbitrary elements of A. Since $(a'x)c' = a'x'c' + a''xc'$ by [1, Lemma 1], $a''xc' = -a'x'c' + (a'x)'c' = c'\{-a'x' + (a'x)'\} = c'a''x$, whence $a''xbc' = c'a''xb = a''xc'b$ follows. Hence $a''A[b, c'] = 0$, and $[b, c'] = 0$. Further, we can easily see that $[b, c''] = 0$. Since $(a'b)'c = c(a'b)'$, by making use of [1, Lemma 1], we can easily see that $a''bc = ca''b = a''cb$. Therefore $a''A[b, c] = 0$, and $c] = 0$.

Proof of Theorem 3. Obviously, every constant in A is in $V_N(A)$; hence if A contains a non-zero constant then $(N, +)$ is abelian, by Lemma 4 (2). Thus, henceforth, we assume that 0 is the only constant in A. Then $A' \neq 0$. Suppose now that $a - a' = 0$ (resp. $a + a' = 0$) for all $a \in A$. Then, for all a, $b \in A$, $ab = (ab)' = ab' + a'b = ab + ab$ (resp. $-ab = (ab)' = ab' + a'b = -ab + a'b$), so $A^2 = 0$ (resp. $A'A = 0$). But this is impossible. We have thus seen that $c - c' \neq 0$ (resp. $c + c' \neq 0$) for some $c \in A$. Now, we can apply Lemma 2 to $u = c - c'$ (resp. $c + c'$) $\in V_A(A)$ and conclude that u is contained in the additive center of A and $u + u = c + c -$

$c' - c'$ (resp. $c + c + c' + c'$) $\in V_N(A)$; hence we get $(N, +)$ abelian, by Lemma 4 (2).

The next is a generalization of [1, Corollary 1].

Corollary 1. *Let N be a distributively-generated prime near-ring. Then each of the conditions* 2), 3) *implies that N is a commutative ring.*

Proof. By Theorem 3, $(N, +)$ is abelian, which in the setting of distributively-generated near-rings forces N to be a ring. Further, it is clear that d is commuting on A ; hence if $d \neq 0$ we can invoke [2, Theorem 1 (2)] to the effect that a prime ring admitting a non-trivial derivation commuting on some non-zero ideal must be commutative. In the event that $d = 0$, Corollary 1 is obvious.

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