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A NOTE ON DERIVATIONS

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L. O. Chung and J. Luh [1] proved the following result : Let R be a semiprime ring with a derivation d. Suppose there exists a positive integer n such that $d(x)^n = 0$ for all $x \in R$ and suppose R is $(n-1)!$-torsion free. Then $d = 0$. A. Giambruno and I. N. Herstein [2] showed that the assumption that R must be $(n-1)!$-torsion free is unnecessary. In Herstein's papers [4] and [5] some related results can be found.

The purpose of this paper is to prove two theorems ; the first one is a generalization of the result of Chung and Luh.

Theorem 1. *Let R be a semiprime ring with a derivation d. Suppose there exist $a \in R$ and a positive integer n such that $ad(x)^n = 0$ for all $x \in R$ (or $d(x)^n a = 0$ for all $x \in R$). If R is $(n-1)!$-torsion free then $ad(x) = 0 = d(x)a$ for all $x \in R$. Moreover, if R is prime, then either $a = 0$ or $d = 0$.*

We will use Theorem 1 in proving the following.

Theorem 2. *Let R be a prime ring of characteristic not 2, and d a nonzero derivation of R. If an additive mapping f of R is such that $f(x)d(x) = 0 = d(x)f(x)$ for all $x \in R$, then $f = 0$.*

For the proof of Theorem 1 we need the lemma below.

Lemma 1 ([1, Lemma 1]). *Let R be a $m!$-torsion free ring. Suppose that $t_1, t_2, \dots, t_m \in R$ satisfy $kt_1 + k^2 t_2 + \cdots + k^m t_m = 0$ for $k = 1, 2, \dots, m$. Then $t_i = 0$ for all i.*

Proof of Theorem 1. We shall consider the case where $d(x)^n a = 0$. The case where $ad(x)^n = 0$ can, of course, be discussed similarly. For the proof we need several steps. We start with the lemma below.

Lemma A. *For all $x, y \in R$, $\sum_{k=0}^{n-1} d(x)^k d(y) d(x)^{n-k-1} a = 0$.* (1)

Proof. A simple modification of the proof of Lemma 2 in [1].

Lemma B. *For all $x, y \in R$, $d^2(x) y d(x)^{n-1} a = 0$.*

Proof. Replacing y by $d(x)y$ in (1) results in

$$\begin{aligned} 0 &= \sum_{k=0}^{n-1} d(x)^k(d^2(x)y+d(x)d(y))d(x)^{n-k-1}a \\ &= \sum_{k=0}^{n-1} d(x)^k d^2(x)yd(x)^{n-k-1}a \\ &\quad + d(x)(\sum_{k=0}^{n-1} d(x)^k d(y)d(x)^{n-k-1}a); \end{aligned}$$

according to (1) the above relation reduces to

$$\sum_{k=0}^{n-1} d(x)^k d^2(x)yd(x)^{n-k-1}a = 0 \text{ for all } x, y \in R. \tag{2}$$

Taking $y = yd(x)^{n-1}$ in (2) and using $d(x)^n a = 0$ one obtains that $d(x)^{n-1}d^2(x)yd(x)^{n-1}a = 0$. The lemma will be proved by showing that $d(x)^{r+1}d^2(x)yd(x)^{n-1}a = 0$, where $r \geq 0$ is any integer, implies $d(x)^r \cdot d^2(x)yd(x)^{n-1}a = 0$. Taking $y = yd(x)^r$ in (2) we get $\sum_{k=0}^{n-1} d(x)^k d^2(x) \cdot yd(x)^{n-k-1+r}a = 0$; since $d(x)^n a = 0$ this relation reduces to

$$d(x)^r d^2(x)yd(x)^{n-1}a + \sum_{k=r+1}^{n-1} d(x)^k d^2(x)yd(x)^{n-k-1+r}a = 0.$$

Hence, if u is an arbitrary element in R, then

$$\begin{aligned} &(d(x)^r d^2(x)yd(x)^{n-1}a)u(d(x)^r d^2(x)yd(x)^{n-1}a) \\ &= -(\sum_{k=r+1}^{n-1} d(x)^k d^2(x)yd(x)^{n-k-1+r}a)u(d(x)^r d^2(x)yd(x)^{n-1}a) \\ &= -\sum_{k=r+1}^{n-1} d(x)^k d^2(x)(yd(x)^{n-k-1+r}aud(x)^r d^2(x)y)d(x)^{n-1}a = 0 \end{aligned}$$

by hypothesis. Since R is semiprime this relation implies that $d(x)^r d^2(x) \cdot yd(x)^{n-1}a = 0$.

Lemma C. *For all* $x, y, z \in R$, $d^2(z)yd(x)^{n-1}a = 0$. (3)

Proof. Take $y \in R$. By Lemma B we have

$$T(x, z) = (d^2(x)+d^2(z))y(d(x)+d(z))^{n-1}a = 0$$

for arbitrary $x, z \in R$. Let us write $(d(x)+d(z))^{n-1}$ as $v_0+v_1+\cdots+v_{n-1}$, where v_j denotes the sum of these terms in which $d(x)$ appears as a factor in the product j times. Since $d^2(x)yd(x)^{n-1}a = d^2(z)yd(z)^{n-1}a = 0$ we have

$$T(x, z) = \sum_{k=0}^{n-2} d^2(x)yv_k a + \sum_{j=1}^{n-1} d^2(z)yv_j a.$$

Thus, if $t_k = d^2(x)yv_{k-1}a + d^2(z)yv_k a$, then we can write $T(x, z) = t_1 + \cdots + t_{n-1}$. Clearly, $T(kx, z) = kt_1 + k^2t_2 + \cdots + k^{n-1}t_{n-1}$ for every integer k. Since $T(kx, z) = 0$, $k = 1, \ldots, n-1$ we have $t_{n-1} = 0$ by Lemma 1. Note that $v_{n-1} = d(x)^{n-1}$. Thus $0 = t_{n-1} = d^2(x)yv_{n-2}a + d^2(z)yd(x)^{n-1}a$. Using this relation and Lemma B, for every $u \in R$ we then have

$$(d^2(z)yd(x)^{n-1}a)u(d^2(z)yd(x)^{n-1}a)$$
$$= -d^2(x)(yv_{n-2}aud^2(z)y)d(x)^{n-1}a = 0.$$

Hence $d^2(z)yd(x)^{n-1}a = 0$ by the semiprimeness of R.

Lemma D. *For all $x \in R$, $d(x)^2a = 0$.*

Proof. Replacing z by x^2 in (3) yields

$$(d^2(x)x + 2d(x)^2 + xd^2(x))yd(x)^{n-1}a = 0.$$

By Lemma B this relation reduces to $2d(x)^2yd(x)^{n-1}a = 0$. Of course, we may assume that $n \geq 3$. Then R is 2-torsion free by assumption and so $d(x)^2yd(x)^{n-1}a = 0$. Since the element y is arbitrary we also have $d(x)^{n-1}ayd(x)^{n-1}a = 0$, hence $d(x)^{n-1}a = 0$ by the semiprimeness of R. Since n is any integer larger than 2 we have by induction $d(x)^2a = 0$.

Lemma E. *For all $x \in R$, $d(x)a = 0$.*

Proof. By Lemma D we may assume that $n = 2$. Hence, by (3) we have $d^2(z)yd(x)a = 0$ for all $x, y, z \in R$. In particular, $d^2(x)ayd^2(x)a = 0$ and also $d^2(z)d(x)ayd^2(z)d(x)a = 0$ which imply

$$d^2(x)a = 0 \text{ for all } x \in R, \tag{4}$$

$$d^2(z)d(x)a = 0 \text{ for all } x, z \in R \tag{5}$$

by the semiprimeness of R. A linearization of $d(x)^2a = 0$ gives

$$d(x)d(y)a + d(y)d(x)a = 0 \text{ for all } x, y \in R. \tag{6}$$

By replacing y by $yd(x)$ in (6) we get $d(x)d(y)d(x)a + d(x)yd^2(x)a + d(y)d(x)^2a + yd^2(x)d(x)a = 0$. Now according to (4), (5) and $d(x)^2a = 0$ this relation reduces to

$$d(x)d(y)d(x)a = 0 \text{ for all } x, y \in R. \tag{7}$$

Linearizing (7) we obtain

$$d(x)d(y)d(z)a + d(z)d(y)d(x)a = 0 \text{ for all } x, y, z \in R. \tag{8}$$

By taking $y = yd(z)$ in (8) we get

$$d(x)d(y)d(z)^2a + d(x)yd^2(z)d(z)a$$
$$+ d(z)d(y)d(z)d(x)a + d(z)yd^2(z)d(x)a = 0.$$

Hence, using (5) and $d(z)^2a=0$ we conclude that $d(z)d(y)d(z)d(x)a=0$. Put $y=yd(x)u$ in this relation. Then we have

$$\begin{aligned} &d(z)d(y)d(x)ud(z)d(x)a+d(z)yd^2(x)ud(z)d(x)a \\ &+d(z)yd(x)d(u)d(z)d(x)a=0 \text{ for all } x, y, z, u\in R. \qquad (9)\end{aligned}$$

By replacing y by $d(u)z$ in (7) we obtain $d(x)d^2(u)zd(x)a+d(x)d(u)d(z)\cdot d(x)a=0$. By (3) this relation reduces to $d(x)d(u)d(z)d(x)a=0$. Thus the last term in (9) is equal to zero. By (3) the second term in (9) is equal to zero as well.

Hence (9) reduces to

$$d(z)d(y)d(x)ud(z)d(x)a=0 \text{ for all } x, y, z, u\in R. \qquad (10)$$

We multiply (6) from the left by $d(y)$ and by (7) it follows that $d(y)^2d(x)a=0$ for all $x, y\in R$. A linearization gives $d(y)d(z)d(x)a+d(z)d(y)d(x)a=0$. Since the element u in (10) is arbitrary we also have $d(z)d(y)d(x)\cdot aud(y)d(z)d(x)a=0$. Combining the last two relations we obtain $d(z)\cdot d(y)d(x)aud(z)d(y)d(x)a=0$ for all $x, y, z, u\in R$. Since R is semiprime this relation implies

$$d(z)d(y)d(x)a=0 \text{ for all } x, y, z\in R. \qquad (11)$$

Substituting xz for z and applying (11), we then get $d(x)zd(y)d(x)a=0$ for all $x, y, z\in R$ which yields $d(y)d(x)a=0$ since R is semiprime. Now, by replacing y by xy we see that $d(x)yd(x)a=0$, hence $d(x)a=0$.

Lemma F. *For all $x\in R$, $ad(x)=0$.*

Proof. By Lemma E we have $0=d(xy)a=d(x)ya+xd(y)a=d(x)ya$. Hence $(ad(x))y(ad(x))=a(d(x)ya)d(x)=0$ and so $ad(x)=0$ since R is semiprime.

From the proof of Lemma F we also see that if R is prime then either $a=0$ or $d(x)=0$ for all $x\in R$. The proof of Theorem 1 is thus completed.

We leave as an open question the following: does Theorem 1 remain true without assuming that R is $(n-1)!$-torsion free?

Our next goal is to prove Theorem 2. First we need two preliminary results. The next lemma is more general than Lemma 3.10 in [3].

Lemma 2. *Let R be a prime ring. If $a, b, c\in R$ are such that $axb=cxa$ for all $x\in R$, then either $a=0$ or $c=b$.*

Proof. In $axb = cxa$ replace x by xay. Then we have $axayb = cxaya$. But $ayb = cya$ and $cxa = axb$, hence we get $ax(c-b)ya = 0$. Since R is prime this gives $a = 0$ or $c = b$.

Proposition 1. *Let R be a prime ring of characteristic not 2, and let d be a nonzero derivation of R. If $a \in R$ is such that $d(x)ad(x) = 0$ for all $x \in R$, then $a = 0$.*

Proof. A linearization of $d(x)ad(x) = 0$ gives $d(x)ad(y)+d(y)ad(x) = 0$. Replacing y by yz we obtain

$$d(x)ad(y)z+d(x)ayd(z)+d(y)zad(x)+yd(z)ad(x) = 0.$$

Since $d(x)ad(y) = -d(y)ad(x)$ and $d(z)ad(x) = -d(x)ad(z)$ we then have $d(y)[z, ad(x)] = [y, d(x)a]d(z)$ where $[u, v]$ denotes the commutator $uv-vu$. Using the last relation we get

$$\begin{aligned} &d(y)[z, ad(x)]y+d(y)z[y, ad(x)] = d(y)[zy, ad(x)] \\ &= [y, d(x)a]d(zy) = [y, d(x)a]d(z)y+[y, d(x)a]zd(y) \\ &= d(y)[z, ad(x)]y+[y, d(x)a]zd(y). \end{aligned}$$

Thus

$$d(y)z[y, ad(x)] = [y, d(x)a]zd(y) \quad \text{for all } x, y, z \in R. \tag{12}$$

Fix $x \in R$. By (12) and Lemma 2 it follows that for every $y \in R$ either $d(y) = 0$ or $[y, ad(x)] = [y, d(x)a]$. In other words, R is the union of its subsets $G = \{y \in R \mid d(y) = 0\}$ and $H = \{y \in R \mid [y, ad(x)-d(x)a] = 0\}$; note that both are additive subgroups of R. But a group cannot be the union of two proper subgroups, hence $G = R$ or $H = R$. Since we have supposed that $d \neq 0$ we are forced to conclude that $H = R$. That is, $[d(x), a]$ is in the center of R for arbitrary $x \in R$. According to $d(x)ad(x) = 0$ we then have

$$d(x)^2a = d(x)[d(x), a] = [d(x), a]d(x) = -ad(x)^2.$$

Multiplying from the left by $d(x)$ we obtain $d(x)^3a = 0$. Now apply Theorem 1. With this the proposition is proved.

Proof of Theorem 2. Linearizing $d(x)f(x) = 0$ we get $d(x)f(y)+d(y)\cdot f(x) = 0$. Multiplying this relation from the right by $d(x)$, since $f(x)d(x) = 0$, it reduces to $d(x)f(y)d(x) = 0$. The result now follows immediately from Proposition 1.

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