Math. J. Okayama Univ. 32 (1990), 77−81

ON WEAKLY PERIODIC RINGS, PERIODIC RINGS AND COMMUTATIVITY THEOREMS

JULIE GROSEN, HISAO TOMINAGA and ADIL YAQUB

Throughout, R will represent a ring with center Z. Let N, E be the set of nilpotent elements of R and the set of idempotents of R, respectively; let N^* be the subset of N consisting of all elements in R which square to zero. For each integer $n>1$, we put $E_n=\{x\in R \mid x^n=x\}$. Call an element x of R *potent* if $x\in P=\bigcup_{n=2}^{\infty} E_n$. A ring R will be called *periodic* if for each $x\in R$, there exist distinct positive integers n, m for which $x^n=x^m$. By [4, Proposition 2], R is periodic if and only if for each $x\in R$, there exists $f(X)\in X^2\mathbf{Z}[X]$ such that $x-f(x)\in N$. If every element of R is expressible as a sum of a potent element and a nilpotent element: $R=P+N$, R is called a *weakly periodic ring*. It is well-known that if R is periodic then it is weakly periodic (see, e.g. [1]). Whether R is weakly periodic implies that R is periodic is apparently not known, except in the presence of additional hypotheses ([2], [3] and [7]).

The major purpose of this paper is to prove the following theorems.

Theorem 1. *Let* $\mathbf{P}$ *be an h-property, that is, a ring-property which is inherited by every subring and every homomorphic image. Then the following statements are equivalent:*

1) *For any weakly periodic ring satisfying* $\mathbf{P}$, *its commutator ideal is nil.*

2) *For every prime* p, $M_2(\mathrm{GF}(p))$ *fails to satisfy* $\mathbf{P}$.

Theorem 2. *Let* R *be a ring. Suppose that* $R=\langle E\cup N\rangle$. *If* N *is an ideal of* R *and* $nE=0$ *for some positive integer* n, *then* R *is periodic.*

Theorem 3. *Let* R *be a weakly periodic ring with* 1, *and let* D_r (*resp.* D_l) *be the set of right* (*resp. left*) *zero-divisors of* R. *Suppose that* N *is commutative and* $D_r\subseteq E+N$. *Then* N *is an ideal of* R *and* $\bar{R}=R/N$ *is either Boolean or a field.*

Theorem 4. *Let* R *be a weakly periodic ring. Suppose that* 1) *for each* $x\in R$ *there exists* $f(X)\in X^2\mathbf{Z}[X]$ *such that* $x-f(x)\in C_R(N)$, *and* 2) *for each* $x\in N+E_n\,(n>1)$ *and* $a\in N$, $[(ax+x)^{n+1}-(xa+x)^{n+1}, x]=0$.

Then R is commutative.

Theorem 5. *Let R be a ring with 1, and $n>1$ a fixed integer. Suppose that $[x-x^n, y-y^n]=0$ for all $x, y \in R$.*

(1) *Let Q be the intersection of the set of non-units of R with the set of quasi-regular elements of R. If $(xy)^n-x^ny^n \in Z$ for all $x, y \in R\backslash Q$, and $(n-1)[a, x]=0$ implies $[a, x]=0$ for all $a \in N$, $x \in R$, then R is commutative.*

(2) *If $(xy)^n-(yx)^n \in Z$ for all $x, y \in R$,* then *R is commutative.*

(3) *If $[x^n, y^n] \in Z$ for all $x, y \in R$, then R is commutative.*

Obviously, Theorem 3 generalizes [5, Theorem 3.3], and Theorem 4 shows that [5, Theorems 2.1, 2.2 and 2.3] are still valid without the hypothesis that R is a periodic ring.

In advance of proving our theorems, we state three lemmas.

Lemma 1. *Let R be a weakly periodic ring. Then the Jacobson radical J of R is nil. If, furthermore, $xR \subseteq N$ for all $x \in N$, then $N=J$ and R is periodic.*

Proof. Let x be an arbitrary element of J, and write $x=b-a$, where $b^n=b\ (n>1)$ and $a \in N$. Then $x+a=b=b^n=(x+a)^n$. Noting that x is in J, we see that $a-a^n \in J$, whence $a \in J$ follows. This proves that $b^n=b=x+a \in J$. Since b^{n-1} is an idempotent with $b=b^{n-1}b$, we get $b=0$, so that $x=-a \in N$. The latter assertion is almost clear.

Lemma 2. *If R is a weakly periodic division ring, then it is a field.*

Proof. Obviously, for each $x \in R$ there exists an integer $n>1$ such that $x^n=x$. Hence R is commutative, by Jacobson's theorem.

Lemma 3. *Suppose that R satisfies the following condition:*

(C) *for each $x, y \in R$ there exist $f(X)$, $g(X)$ in $X^2\mathbf{Z}[X]$ such that $[x-f(x), y-g(y)]=0$.*

If for each $a \in N^*$ and $x \in R$, there exists a positive integer k such that $[a, x]_k\ (=[[a, x]_{k-1}, x])=0$, then R is commutative.

Proof. By [6, Theorem C and Lemma 1 (2)], we see that $[a, x]=0$ for all $a \in N^*$ and $x \in R$. Hence, [6, Lemma 2] shows that $R=C_R(N^*)$ is commutative.

Proof of Theorem 1. Since $M_2(\mathrm{GF}(p))$ is a periodic ring, it remains only to prove that 2) implies 1). Let J be the Jacobson radical of a weakly periodic ring R satisfying **P**, and let R' be a primitive homomorphic image of R. If R' is not a division ring then, by the structure theorem of primitive rings, we can easily see that there exists a prime p such that $M_2(\mathrm{GF}(p))$ is a factorsubring of R'. But this contradicts 2). Hence R' has to be a division ring, so R' is a field by Lemma 2. We have thus seen that R/J is commutative, that is the commutator ideal $C(R)$ of R is contained in J. Hence, by Lemma 1, $C(R)$ is nil, which proves the theorem.

Corollary 1. *Let* **P** *be an h-property. Suppose that for each prime p,* $M_2(\mathrm{GF}(p))$ *fails to satisfy* **P**. *Then every weakly periodic ring satisfying* **P** *is periodic.*

Corollary 2. *Let R be a weakly periodic ring, and $m > 1$ an integer. Suppose that for each $x_1, \dots, x_m \in R$, there exists a monic monomial (word) w and a polynomial f in $\mathbf{Z}\langle X_1, \dots, X_m\rangle$ such that*

$$[x_1 w(x_1, \dots, x_m) x_m - x_m f(x_1, \dots, x_m) x_1, x_1] = 0.$$

Then R is periodic.

Proof of Theorem 2. As is well-known, R/N is a subdirect sum of subdirectly irreducible rings R_i $(i \in I)$. Since R/N is generated by central idempotents as ring, R_i is generated by an identity element as ring. Noting that $nE = 0$, we see that R_i is a homomorphic image of $\mathbf{Z}/n\mathbf{Z}$. It is easy to see that $\mathbf{Z}/n\mathbf{Z}$ satisfies the polynomial identity $X^{2k} - X^k = 0$ for some positive integer k. Also R/N satisfies the same identity. Thus, R is periodic by [4, Proposition 2].

Corollary 3. *Suppose that $R = \langle E \cup N \rangle$ and $nE = 0$ for some positive integer n. If N is commutative then R is periodic. In particular, if R satisfies the condition* (C) *then R is periodic.*

Proof. By [2, Theorem 2], N is a commutative ideal. The latter assertion is clear by [6, Theorem C].

Proof of Theorem 3. In view of [2, Theorem 2], N is an ideal of R and R is periodic by [4, Proposition 2]. As is easily seen, $D_r = D_l = R \setminus U$, where U is the set of units in R. Hence $\bar{R} = \bar{E} \cup \bar{U}$ is commutative, by Jacobson's theorem. Suppose now that there exists an idempotent $\bar{e} \neq 0, 1$ in $\bar{R}$. Then, for each $\bar{x} \in \bar{R}$, $\bar{e}\bar{x} \notin \bar{U}$ and $(1-\bar{e})\bar{x} \notin \bar{U}$. Hence $\bar{e}\bar{x}^2 = (\bar{e}\bar{x})^2 = \bar{e}\bar{x}$ and $(1-\bar{e})\bar{x}^2 = (1-\bar{e})\bar{x}$, whence $\bar{x}^2 = \bar{x}$ follows.

Proof of Theorem 4. By 1), we can easily see that N is commutative. Hence, by [2, Theorem 2], N is a commutative ideal.

Now, let $x \in N+E_n$ $(n>1)$ and $a \in N$. Then $x-x^n \in N$ and $x^2-x^{n+1} \in N$. In particular, R satisfies the condition (C). Further, noting that $N^2 \subseteq Z$, we have

$$[[a, x^{n+1}], x] = [(ax+x)^{n+1}-(xa+x)^{n+1}, x] = 0.$$

Combining this with $[a, x^2-x^{n+1}] = 0$, we get $[[a, x^2], x] = 0$. Hence, by [6, Lemma 1 (2)],

$$[a, x^2] = 0 \text{ for all } x \in R \text{ and } a \in N.$$

In particular, this proves that $E \subseteq Z$.

The usual argument then shows that we may assume, without loss of generality, that R is a local ring with radical N and characteristic p^α for some prime p (see [1, Lemma 1 (d)]). In order to see that R is commutative, it is enough to show that $[a, x] = 0$ for all $x \in R$ and $a \in N$ (see Lemma 3). Obviously, $2[a, x] = [a, (x+1)^2] = 0$. In case $p \neq 2$, $[a, x] = 0$ is immediate. Henceforth, we assume $p=2$. Further, we may assume that $x \notin N$. Then, by $x^n - x \in N$ with some $n>1$, $\bar{x} = x+N$ generates a finite field $\mathrm{GF}(2^k)$. Hence $[a, x^{2^k}-x] = 0$, and therefore $[a, x] = [a, x^{2^k}] = [a, (x^2)^{2^{k-1}}] = 0$.

Proof of Theorem 5. In view of Lemma 3, it is enough to show that $[a, x]_3 = 0$ for all $a \in N^*$ and $x \in R$. By [6, Theorem C], N is a commutative ideal, so $N^2 \subseteq Z$. If $a \in N$ and $x \in R$, then $[a, x-x^n] = [a-a^n, x-x^n] = 0$, and hence $[a, x] = [a, x^n]$.

(1) Let $a \in N^*$. If $x \in R \backslash Q$ then, since $N^2 \subseteq Z$, $(n-1)[a, x]_2 = [(n-1)[a, x^n], x] = [(x(1+a))^n - x^n(1+a)^n, x] - [((1+a)x)^n - (1+a)^n x^n, x] = 0$, and therefore $[a, x]_2 = 0$. If $x \in Q$, then $1-x \in R \backslash Q$, so $[a, x]_2 = [a, 1-x]_2 = 0$, by the above.

(2) Let $a \in N^*$, $x \in R$. Then, since $N^2 \subseteq Z$, $[a, x]_2 = [[a, x^n], x] = [((1+a)x)^n - (x(1+a))^n, x] = 0$.

(3) Let $a \in N^*$, $x \in R$. Then, since $N^2 \subseteq Z$, $[a, x]_3 = [[a, x], x^n]_2 = [[a, x^n], x^n]_2 = [[(x+ax)^n, x^n] - [(x+xa)^n, x^n], x^n] = 0$.

REFERENCES

[1] H. E. BELL : A commutativity study for periodic rings, Pacific J. Math. 70 (1977), 29−36.

[2] H. E. BELL and H. TOMINAGA : On periodic rings and related rings, Math. J. Okayama Univ. 28 (1986), 101−103.

[3] H. E. BELL and A. YAQUB : Some periodic conditions for rings, Math. J. Okayama Univ. 29 (1987), 179−183.

[4] M. CHACRON : On a theorem of Herstein, Canad. J. Math. 21 (1969), 1348−1353.

[5] J. GROSEN : Rings satisfying polynomial identities or constraints on certain subsets, Thesis, University of California, Santa Barbara, 1988.

[6] H. KOMATSU and H. TOMINAGA : Chacron's condition and commutativity theorems, Math. J. Okayama Univ. 31 (1989), 101−120.

[7] M. ÔHORI : On strongly π-regular rings and periodic rings, Math. J. Okayama Univ. 27 (1985), 49−52.

J. GROSEN AND A. YAQUB
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF CALIFORNIA
SANTA BARBARA, CALIFORNIA 93106, U. S. A.

H. TOMINAGA
DEPARTMENT OF MATHEMATICS
OKAYAMA UNIVERSITY
OKAYAMA, 700 JAPAN

(*Received April 12, 1989*)