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A NOTE ON ZERO COMMUTATIVE AND DUO RINGS

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In this paper we show that artinian duo rings are (finite) direct sum of local subrings and artinian duo QF-3 rings are actually QF-rings and hence exact rings in the sense of Azumaya [2] and have the self-duality.

A ring R is a left (resp. right) duo ring if every left (resp. right) ideal is a two-sided ideal. R is duo if it is both left and right duo. R is zero commutative (zc) if for all $a, b \in R$, $ab = 0$ implies $ba = 0$. R is zero insertive (zi) if for all $a, b, x \in R$, $ab = 0$ implies $axb = 0$.

It can be shown that every zc ring is zi but the converse is not true ([5]). Every duo ring is zi. The idempotents in a zi ring are central idempotents.

For $a \in R$, let $\ell(a) = \{r \in R \mid ra = 0\}$, i.e., the left annihilator of a in R and $r(a) = \{r \in R \mid ar = 0\}$, i.e., the right annihilator of a in R.

Lemma 1 ([5]). *For any ring R the following are equivalent:*

(1) *R is zi.*

(2) *For each $a \in R$, $\ell(a)$ (equivalently $r(a)$) is a two-sided ideal.*

Proof. (1) ⇒ (2) Suppose that R is zi and let $a, s \in R$ and $r \in \ell(a)$. Then $ra = 0$ and so $rsa = 0$, which implies $rs \in \ell(a)$. So $\ell(a)$ is a right ideal, but it is known to be a left ideal. So it is two-sided. Reversing the proof shows (2) ⇒ (1).

Lemma 2. *Let e be an idempotent of R. Then the following conditions are equivalent:*

(1) *e is in the center of R.*

(2) *$Re = eR$.*

(3) *$\ell(e) = r(e)$.*

Proof. (1) ⇒ (2) and (1) ⇒ (3) are clear. Now assume (2). Let $r \in R$. Then $re \in eR$ implies $ere = re$. Similarly, $ere = er$ so $re = er$. Thus e is in the center of R, and this shows (2) ⇒ (1). Assume (3). Since e is an idempotent, $\ell(e) = R(1-e)$ and $r(e) = (1-e)R$. Therefore (3) implies

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$R(1-e) = (1-e)R$. Applying then the implication (2) ⇒ (1) to $1-e$ instead of e, we know that $1-e$ is and hence e is in the center of R.

For a ring R let J denote the Jacobson radical of R, i.e., the intersection of all maximal left ideals of R.

Proposition 3. *Let R be a left artinian ring. Then the following conditions are equivalent:*

(1) *Every idempotent of R is in the center of R.*

(2) *R is a direct sum of (orthogonal) local subrings.*

Proof. Let e be a primitive idempotent of R. Then as is well-known, Je is a unique maximal left subideal of Re. Assume e is in the center of R. Then e is the identity element of the subring $eRe = eR = Re$. Moreover since $ra = rea$ for all $r \in R$ and $a \in eRe$, every left ideal of the ring $eRe = Re$ is a left ideal of R. Thus Je ($= eJ$) is a unique maximal left ideal of eRe, and therefore eRe is a local ring ([1]). Let $1 = e_1 + e_2 + \cdots + e_n$ be a decomposition of 1 as a sum of orthogonal primitive idempotents e_i, which are all in the center of R. Then

$$R = Re_1 \oplus Re_2 \oplus \cdots \oplus Re_n = e_1Re_1 \oplus e_2Re_2 \oplus \cdots \oplus e_nRe_n$$

gives a direct decomposition of R into orthogonal local subrings. Thus (1) ⇒ (2) is proved.

Assume conversely (2), i.e., $R = R_1 \oplus R_2 \oplus \cdots \oplus R_n$ where R_i's are orthogonal local subrings of R. Let $1 = e_1 + e_2 + \cdots + e_n$, $e_i \in R_i$ be the corresponding decomposition of 1. Thus, as is well-known, $e_1, e_2, \dots, e_n$ are orthogonal idempotents in the center of R and each e_i is the identity element of $R_i = e_iRe_i$. Let e be any idempotent of R. Then, since each e_i is in the center, $ee_i (= e_ie)$ is an idempotent contained in R_i. Since R_i is a local ring, $ee_i = 0$ or $ee_i = e_i$ and so ee_i is any way in the center of R. Since further $e = ee_1 + ee_2 + \cdots + ee_n$ we know that e is in the center of R. This proves (2) ⇒ (1).

Corollary 4. *Every artinian duo, every zc and every zi ring is a direct sum of local subrings.*

Proof. Every duo ring is *zi* ring and every *zc* ring is *zi* ring and in a *zi* ring all idempotents are central. We now apply the above proposition.

Lemma 5. *Let R be a ring with all the idempotents are central. Then*

$ab = 1$ *implies* $ba = 1$ *for all* $a, b \in R$.

Proof. Let a and b be non-zero in R with $ab = 1$. Then $(ba)^2 = baba = ba$ is an idempotent. Since it is central, we have $b\,(ba) = (ba)\,b = b\,(ab) = b$. Now $ba = 1\,(ba) = (ab)\,(ba) = ab^2a = ab = 1$.

Lemma 5 shows that if R is a *zi* ring, then R is "1" commutative. In particular this is the case for every *zc* and every duo ring.

A left artinian ring R is a Quasi-Frobenius (QF-) ring if $\ell(r(L)) = L$ and $r(\ell(K)) = K$ for every left ideal L and every right ideal K of R. This is equivalent to the conditions that R is left artinian and R is injective as a left R-module. A ring R is a QF-3 ring if it has a unique minimal faithful left R-module. This is equivalent to the condition that the injective envelope $E(I)$ of every simple left ideal I of R is isomorphic to Re for some primitive idempotent e in R.

Proposition 6. *A QF-3 left artinian ring R is zi if and only if R is duo.*

Proof. Let R be a left artinian QF-3 *zi* ring. By Corollary 4 we may assume that R is a local ring. Let I be a simple left ideal of R. Then the injective envelope $E(I)$ of I is isomorphic to Re for some primitive idempotent e in R([1]). But R, being local, has 1 as the only primitive idempotent. So $E(I) \cong R$, i.e., R is injective thus R is actually QF ring.

Now let L be a left ideal of R. Then $\ell(r(L)) = L$. Since R is *zi*, $\ell(r(L))$ is a two-sided ideal by Lemma 1. Similarly every right ideal of R is two-sided, and thus R is duo.

It is well-know that every artinian QF ring has the self-duality. By [4] artinian duo rings are exact rings in the sense of Azumaya [2], so we have another example of an exact ring with self-duality.

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