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PRIME IDEALS OF SKEW POLYNOMIAL RINGS AND SKEW LAURENT POLYNOMIAL RINGS

Dedicated to Professor Takasi NAGAHARA on his 60 th birthday

EDUARDO CISNEROS, MIGUEL FERRERO and MARIA INÉS CONZÁLEZ

0. Introduction. Let R be a ring and let $R[X]$ be the polynomial ring over R. The structure of R-disjoint ideals of $R[X]$ has been studied in [3]. In particular, we have a complete description of the prime ideals of $R[X]$ and a one-to-one correspondence between the set of R-disjoint prime ideals of $R[X]$, the set of Q-disjoint prime ideals of $Q[X]$ and the set of monic irreducible polynomials of $C[X]$, where Q is a ring of right quotients of R and C is the extended centroid of R. For a skew polynomial ring of derivation type $R[X; d]$, where d is a derivation of R, the corresponding matter has been considered in [6].

Now, let ρ be an automorphism of the ring R. The skew Laurent polynomial ring $R\langle X; \rho\rangle$ is the ring whose elements are of the form $\sum_{i=-n}^{n} X^i b_i$, $b_i \in R$, where the addition is defined as usually and the multiplication by $bX = X\rho(b)$, for all $b \in R$ [4]. The skew polynomial ring $R[X; \rho]$ is the subring of $R\langle X; \rho\rangle$ whose elements are the polynomials $\sum_{i=0}^{n} X^i b_i$, $b_i \in R$. The purpose of this paper is to study prime ideals of $R\langle X; \rho\rangle$ and $R[X; \rho]$.

We use § 1 as an introductory section. In § 2 we study R-disjoint prime ideals of $R\langle X; \rho\rangle$. The main result states that if P is an R-disjoint ideal of $R\langle X; \rho\rangle$ then P is prime if and only if R is ρ-prime and $P = f_0 Q\langle X; \rho\rangle \cap R\langle X; \rho\rangle$, where Q is the ρ-quotient ring of R and f_0 is an irreducible polynomial of the center of $Q\langle X; \rho\rangle$. This result extends the results of [3]. We also give an intrinsic characterization for P to be a prime ideal.

In § 3 we study prime ideals of $R[X; \rho]$. We prove that there is a one-to-one correspondence between the set of all R-disjoint prime ideals P of $R[X; \rho]$ with $X \notin P$ and the set of all R-disjoint prime ideals of $R\langle X; \rho\rangle$. Then we have a description of those prime ideals using the results of the former section.

Finally, in § 4, we apply the results to get necessary and sufficient

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conditions for every prime ideal of $R\langle X;\rho\rangle$ ($R[X;\rho]$) to be nonsingular.

1. **Prerequisites.** Throughout this paper every ring has an identity element. If R is a ring and ρ is an automorphism of R, then an ideal I of R is said to be a ρ-ideal (ρ-invariant ideal) if $\rho(I)\subseteq I$ ($\rho(I)=I$). Let P be a ρ-invariant ideal of R (we denote it by $P\lhd_\rho R$). Then P is said to be ρ-prime (strongly ρ-prime) if $IJ\subseteq P$ for any ρ-invariant ideals I and J (ρ-ideal I and ideal J) of R implies either $I\subseteq P$ or $J\subseteq P$. The ring R is said to be ρ-prime (strongly ρ-prime) if the ideal (0) of R is ρ-prime (strongly ρ-prime). Clearly, if R is strongly ρ-prime then R is ρ-prime. Our terminology is taken from [1] and does not agree with that of references [10] and [11]. It is also convenient to remark that strongly ρ-prime is not the same as ρ-strongly prime (see [5]).

Let R be a ρ-prime ring. As in ([9], Ch. 3) we define the right (Martindale) ρ-quotient ring Q of R as $\varinjlim_{I\in\mathscr{F}}\mathrm{Hom}_R(I_R,R_R)$, where $\mathscr{F}$ is the filter of all non-zero ρ-invariant ideals of R. By C we denote the center of Q. The automorphism ρ can be extended to a unique automorphism of Q which we will denote by ρ again and we put $C_\rho=\{a\in C:\rho(a)=a\}$. The ring Q inherits all basic properties of the classical Martindale's construction. In particular, we easily have the following (c.f [2], Lemma 1.2).

Lemma 1.1. (i) $R\subseteq Q$.

(ii) *If* $0\neq I\lhd_\rho R$ *and* $f:I\to R$ *is a homomorphism of right* R-*modules, then there exists* $q\in Q$ *such that* $f(r)=qr$, *for all* $r\in I$. *Moreover,* $q\in C$ *if and only if* f *is an* R-*bimodule homomorphism.*

(iii) *For any* $q_1,\ldots,q$ *in* Q *there exists* $0\neq I\lhd_\rho R$ *such that* $q_iI\subseteq R$ *for* $i=1,\ldots,n$.

(iv) *If* $qI=0$ *for some* $q\in Q$ *and* $0\neq I\lhd_\rho R$, *then* $q=0$.

(v) Q *is* ρ-*prime.*

We will need also the following.

Lemma 1.2. *Assume that* $q\in Q$ *verifies* $qR=Rq$ *and* $\rho(q)=q$. *Then* q *is invertible in* Q. *In particular,* C_ρ *is a field.*

Proof. $I=qR\cap R$ is a ρ-invariant ideal of R. If $qr=0$, for some $r\in R$, then $Ir=0$ and so $r=0$. Hence the map $f:I\to R$ defined by $f(qr)=r$ is a (well defined) right homomorphism. Then the element of Q corresponding to f is an inverse of q.

When R is prime, the center $Z(Q[X;\rho])$ of $Q[X;\rho]$ has been described in ([8], Proposition 2.3). Repeating the arguments in [8] we can prove the following lemma.

Firstly, suppose that ρ^k is an inner automorphism of Q for some $k \geq 1$. Then there exists the smallest non-zero natural number m such that ρ^m is an inner automorphism of Q determined by a ρ-invariant element $b \in Q$. We have:

Lemma 1.3. (i) *If ρ^k is not an inner automorphism of Q for all $k \geq 1$, then $Z(Q\langle X;\rho\rangle) = Z(Q[X;\rho]) = C_\rho$.*

(ii) *If ρ^k is an inner automorphism of Q for some $k \geq 1$, then $Z(Q\langle X;\rho\rangle) = C_\rho\langle z\rangle$ and $Z(Q[X;\rho]) = C_\rho[z]$, where $z = X^m b^{-1}$, m and b as above.*

The automorphism ρ of R can be extended to an automorphism of $R\langle X;\rho\rangle$ (and $R[X;\rho]$) by the natural way. We denote the extension by ρ again. If I is an ideal of $R\langle X;\rho\rangle$, then I is a ρ-invariant ideal. We say that I is R-disjoint if $I \cap R = 0$.

An element of $R[X;\rho]$ is called a polynomial and a proper polynomial if its constant term is non-zero. In case that f is a proper polynomial, the degree of f and the leading coefficient of f are defined in the obvious manner and denoted by ∂f and $\ell c(f)$, respectively.

If I is a non-zero R-disjoint ideal of $R\langle X;\rho\rangle$, there exists a proper polynomial of minimal degree n in I. The integer n is said to be the minimality of I and denoted by $\mathrm{Min}(I)$. We denote by $\tau(I)$ the ρ-invariant ideal of R of all the leading coefficients of proper polynomials of minimal degree in I (together with 0).

2. Prime ideals of R⟨X; ρ⟩. If P is a prime ideal of $R\langle X;\rho\rangle$, then $P \cap R$ is a ρ-prime ideal of R. By factoring out $P \cap R$ and $(P \cap R)\langle X;\rho\rangle$ from R and $R\langle X;\rho\rangle$, respectively, we may assume that R is ρ-prime and P is R-disjoint. So, throughout this section we assume that R is ρ-prime. We denote by Q the (right) ρ-quotient ring of R and by Z the center of $Q\langle X;\rho\rangle$.

We begin with the following.

Lemma 2.1. *Let I be a non-zero R-disjoint ideal of $R\langle X;\rho\rangle$ with $\mathrm{Min}(I) = n$. Then there exists a unique monic proper polynomial $f_I \in$*

$Q\langle X;\rho\rangle$ such that for any polynomial $f\in I$ with $\partial f=n$ we have $f=f_I\ell c(f)$. In addition, $X^{-n}f_I\in Z$.

Proof. If $a\in\tau(I)$, then there exists a unique

$$f=X^n a+X^{n-1}a_{n-1}+\cdots+a_0\in I.$$

Therefore the map $\alpha_i\colon\tau(I)\to R$ defined by $\alpha_i(a)=a_i$ is a (well defined) homomorphism of right R-modules, $i=0,1,\dots,n$ (where $a_n=a$). Since $\tau(I)$ is a ρ-invariant ideal there are elements $q_n=1, q_{n-1},\dots,q_0$ in Q such that $q_i a=a_i$, $i=0,1,\dots,n$. Define $f_I=X^n+X^{n-1}q_{n-1}+\cdots+q_0\in Q\langle X;\rho\rangle$, which is clearly the unique proper polynomial such that $f=f_I\ell c(f)$, for any polynomial $f\in I$ with $\partial f=n$.

Now we show that $X^{-n}f_I\in Z$. For any $a\in\tau(I)$, $\rho(a)\in\tau(I)$. Thus $f_I a\in I$ and $f_I\rho(a)\in I$. Hence $(f_I-\rho(f_I))\rho(a)=f_I\rho(a)-\rho(f_I a)\in I$ and since $\partial(f_I-\rho(f_I))<n$, we have $(f_I-\rho(f_I))\tau(I)=0$. This implies that $\rho(f_I)=f_I$ and so $Xf_I=f_IX$. Also, for any $a\in\tau(I)$ and $b\in R$, $bf_Ia\in I$ and $f_I\rho^n(b)a\in I$. Since $\partial(bf_I-f_I\rho^n(b))<n$ it follows as above that $bf_I=f_I\rho^n(b)$. Now we can get easily the required relation.

The polynomial f_I constructed in the above lemma will be called the canonical polynomial of the non-zero R-disjoint ideal I.

Corollary 2.2. *Let f_I be the canonical polynomial of the R-disjoint ideal I of $R\langle X;\rho\rangle$. Then $I\subseteq f_IQ\langle X;\rho\rangle\cap R\langle X;\rho\rangle$.*

Proof. Suppose $f\in I$ is a polynomial. Since f_I is monic there exist polynomials h and r in $Q\langle X;\rho\rangle$ such that $f=f_Ih+r$, where either $r=0$ or $\partial r<\partial f_I=\mathrm{Min}(I)$. Take a non-zero ρ-invariant ideal J of R such that hJ and rJ are contained in $R\langle X;\rho\rangle$. We get easily $r\tau(I)J\subseteq I$ and so $r\tau(I)J=0$. Since $\tau(I)J\neq 0$ it follows that $r=0$, i.e., $f=f_Ih\in f_IQ\langle X;\rho\rangle\cap R\langle X;\rho\rangle$.

Now, if f is an arbitrary element of I, there exists an integer $t\geq 0$ such that $X^tf\in I$ is a polynomial. Then $X^tf\in f_IQ\langle X;\rho\rangle$ and so $f\in X^{-t}f_IQ\langle X;\rho\rangle\cap R\langle X;\rho\rangle=f_IQ\langle X;\rho\rangle\cap R\langle X;\rho\rangle$.

Let I be a non-zero R-disjoint ideal of $R\langle X;\rho\rangle$ and let f_I be the canonical polynomial of I. Since $X^{-\partial(f_I)}f_I\in Z$ it follows that $f_IQ\langle X;\rho\rangle$ is an ideal of $Q\langle X;\rho\rangle$. We define the closure $[I]$ of I by $[I]=f_IQ\langle X;\rho\rangle\cap R\langle X;\rho\rangle$. The ideal I is said to be closed if $[I]=I$.

It is convenient to have an intrinsic characterization of a closed ideal.

Firstly, if I is an R-disjoint ideal of $R\langle X;\rho\rangle$ and $f=X^n a+X^{n-1}a_{n-1}+\cdots+a_0$ is a proper polynomial of minimal degree n in I, then $g=ar\rho^j(f)-\rho^n(fr)\rho^j(a)\in I$ $(r\in R,\ j\in \mathbf{Z})$ and $\partial g<n$. So we have

$$(*):\ ar\rho^j(f)=\rho^n(fr)\rho^j(a),\ \text{for all } r\in R,\ j\in \mathbf{Z}.$$

Now, let Γ_R be the set of all proper polynomials in $R\langle X;\rho\rangle$ which satisfy the condition $(*)$. For $f\in\Gamma_R$ with $\ell c(f)=a$ we put

$$[f]_R=\{g\in R\langle X;\rho\rangle:\ \text{there is } 0\neq J\lhd_\rho R \text{ such that } \rho^i(g)Ja\subseteq R\langle X;\rho\rangle f,\ \text{for all } i\in\mathbf{Z}\}.$$

Hereafter we denote Γ_R and $[f]_R$ simply by Γ and $[f]$ and we use Γ_Q and $[f]_Q$ for the corresponding subsets of $Q\langle X;\rho\rangle$. Note that $\Gamma_Q^0=\{f_0\in\Gamma_Q: f_0 \text{ is monic}\}$ is equal to the set of all the monic proper polynomials g of $Q\langle X;\rho\rangle$ such that $X^{-\partial g}g\in Z$. In particular, if I is an R-disjoint ideal of $R\langle X;\rho\rangle$, then the canonical polynomial f_I of I is in Γ_Q^0.

Lemma 2.3. *If $f\in\Gamma$, then $[f]$ is an R-disjoint ideal of $R\langle X;\rho\rangle$ which contains f as a proper polynomial of minimal degree.*

Proof. Write $f=X^n a+\cdots+a_0$. It is easy to see that $[f]$ is an ideal of $R\langle X;\rho\rangle$. Also, by condition $(*)$, $\rho^i(f)ra=\rho^{i-n}(a)\rho^{-n}(r)f\in R\langle X;\rho\rangle f$, for all $r\in R$, $i\in\mathbf{Z}$. Then $f\in[f]$.

Suppose there exists a proper polynomial $0\neq h\in[f]$ with $\partial h<\partial f$. Then there exists a non-zero ρ-invariant ideal J of R with $hJa\subseteq R\langle X;\rho\rangle f$. Take $b\in J$ such that $hba\neq 0$. Then $hba=gf$, for some $g=X^m b_m+\cdots+X^s b_s$ $(s<m)$, and we may assume that g is chosen with $m-s$ being minimal. If $m\geq 0$, then $\rho^n(b_m)a=0$. Using $(*)$ and the ρ-primeness of R we easily get $b_m f=0$. Thus $gf=(g-X^m b_m)f$. Hence we may assume that $m<0$. In this case we have $b_s a_0=0$. Again, using $(*)$, we get $b_s f=0$ and so $gf=(g-X^s b_s)f$, a contradiction.

Proposition 2.4. *Let I be an R-disjoint ideal of $R\langle X;\rho\rangle$ and let f be any polynomial of minimal degree n in I. Then $f\in\Gamma$ and $[I]=[f]$.*

Proof. We have already seen that $f\in\Gamma$. Let f_I be the canonical polynomial of I. Then $f=f_I a$, where $a=\ell c(f)$. Suppose $h=f_I g\in I$, $g\in Q\langle X;\rho\rangle$, and let J be a non-zero ρ-invariant ideal of R with $gJ\subseteq R\langle X;\rho\rangle$. Hence $\rho^i(h)Ja=f_I\rho^i(g)Ja=X^n\rho^i(gJ)X^{-n}f_I a\subseteq R\langle X;\rho\rangle f$, for every $i\in\mathbf{Z}$, and it follows that $h\in[f]$. Consequently $[I]\subseteq[f]$.

Conversely, suppose $g \in [f]$ and let L be a non-zero ρ-invariant ideal of R such that $\rho^i(g)La \subseteq R\langle X;\rho\rangle f$, for all $i \in \mathbf{Z}$. There exists $t \geq 0$ with $X^t g \in R[X;\rho]$. Since f_I is monic there exist h and r in $Q[X;\rho]$ such that $X^t g = f_I h + r$, where either $r = 0$ or $\partial r < n$. We easily get $\rho^i(r)La \subseteq f_I Q[X;\rho]$ for every $i \in \mathbf{Z}$, hence $\rho^i(r)La = 0$ and so $r = 0$. Thus $g = f_I X^{-t} h \in [I]$ and the proof is complete.

Corollary 2.5. *Let I be a non-zero R-disjoint ideal of $R\langle X;\rho\rangle$. Then $[I]$ is the largest ideal H of $R\langle X;\rho\rangle$ which contains I and satisfies* Min$(H) =$ Min(I). *In particular*, $[[I]] = [I]$.

Proof. It is clear that Min$([I]) =$ Min(I). If $H \supseteq I$ and Min$(H) =$ Min(I), choose a polynomial f of minimal degree in I. Then $H \subseteq [H] = [f] = [I]$.

Next we will need the following

Lemma 2.6. *A Q-disjoint ideal J of $Q\langle X;\rho\rangle$ is closed if and only if $J = f_0 Q\langle X;\rho\rangle$ for some monic proper polynomial $f_0 \in \Gamma_Q$.*

Proof. Suppose that $f_0 \in \Gamma_Q^0$ and $n = \partial(f_0)$. Since $X^{-n}f_0 \in Z$ is clear that $f_0 Q\langle X;\rho\rangle$ is an ideal of $Q\langle X;\rho\rangle$. Let I be an ideal of $Q\langle X;\rho\rangle$ such that $I \supseteq f_0 Q\langle X;\rho\rangle$ and Min$(I) = n$. If $g \in I$, using the division argument we get $g = f_0 h$, $h \in Q\langle X;\rho\rangle$. Consequently, $I = f_0 Q\langle X;\rho\rangle$ is closed by Corollary 2.5.

Conversely, assume that J is a closed ideal of $Q\langle X;\rho\rangle$. Consider the non-zero R-disjoint ideal $I = J \cap R\langle X;\rho\rangle$ of $R\langle X;\rho\rangle$ and the canonical polynomial f_I. It is clear that $\partial(f_I) =$ Min$(I) =$ Min(J) and we can easily see that $J = f_I Q\langle X;\rho\rangle$.

Before proceeding to apply the former results to study prime ideals we recall the following.

Lemma 2.7 (*c.f.* [1], *Lemma* 1.4 *and Proposition* 1.6). *Let P be a non-zero R-disjoint ideal of $R\langle X;\rho\rangle$. Then P is prime if and only if R is ρ-prime and P is maximal with respect to $P \cap R = 0$.*

Let f be a proper polynomial in Γ_R. We say that f is irreducible in Γ_R when the following condition holds: if there exist $g \in \Gamma_R$ and a proper polynomial $h \in R\langle X;\rho\rangle$ such that $f = gh$, then $\partial g = \partial f$. Similarly, we define the irreducibility of a proper polynomial in Γ_Q.

Now we can prove the main result of this section.

Theorem 2.8. *Let R be a ρ-prime ring and P a non-zero R-disjoint ideal of $R\langle X;\rho\rangle$. Then the following conditions are equivalent.*

(i) *P is prime.*

(ii) *P is closed and every $f\in P$ with $\partial f=\mathrm{Min}(P)$ is irreducible in Γ_R.*

(iii) *$P=f_0Q\langle X;\rho\rangle\cap R\langle X;\rho\rangle$, where f_0 is a monic proper polynomial in Γ_Q which is irreducible in Γ_Q.*

Proof. (i) → (ii). P is closed by Lemma 2.7. Suppose $f\in P$ and $\partial f=\mathrm{Min}(P)$. If $f=gh$, $g\in\Gamma_R$, then $f\in gR\langle X;\rho\rangle\subseteq[g]$. It is easy to see that this implies $[f]\subseteq[g]$. Then $P=[f]=[g]$ and so $\partial g=\partial f$. Thus f is irreducible in Γ_R.

(ii) → (iii). If P is closed then $P=f_PQ\langle X;\rho\rangle\cap R\langle X;\rho\rangle$, for $f_P\in\Gamma_Q^0$. Suppose $f_P=gh$, where $g\in\Gamma_Q$. Let J be a non-zero ρ-invariant ideal of R with $gJ\subseteq R\langle X;\rho\rangle$ and $hJ\subseteq R\langle X;\rho\rangle$. Put $n=\partial f_P$ and $s=\partial g$ and choose b_1,b_2 in J such that $a=\rho^n(b_1)b_2\neq 0$. We have $f_Pa=b_1f_Pb_2=b_1ghb_2=g\rho^s(b_1)hb_2\in P$, $\partial(f_Pa)=\mathrm{Min}(P)$ and $g\rho^s(b_1)\in\Gamma_R$. Hence $\partial g=\partial(g\rho^s(b_1))=\partial(f_Pa)=\partial f_P$. Consequently, f_P is irreducible in Γ_Q.

(iii) → (i). Let L be an R-disjoint ideal of $R\langle X;\rho\rangle$ with $L\supseteq P$. Replacing L by $[L]$ we may assume that L is closed, i.e., $L=h_0Q\langle X;\rho\rangle\cap R\langle X;\rho\rangle$ for some $h_0\in\Gamma_Q^0$. If $f_0=h_0g+r$, where either $r=0$ or $\partial r<\partial h_0$, we easily get $r=0$. Then $f_0=h_0g$ and irreducibility gives $\partial h_0=\partial f_0$, so $h_0=f_0$. Consequently, $P=L$ and P is prime by Lemma 2.7.

If there exists a non-zero R-disjoint ideal of $R\langle X;\rho\rangle$, then $Z\neq C_\rho$ by Lemma 2.1. Hence we know that $Z=C_\rho\langle z\rangle$, where $z=X^mb^{-1}$, m and b^{-1} as in Lemma 1.3. Using this notation we have.

Corollary 2.9. *Let P be a non-zero R-disjoint ideal of $R\langle X;\rho\rangle$. Then the following conditions are equivalent.*

(i) *P is prime.*

(ii) *$P=g_0Q\langle X;\rho\rangle\cap R\langle X;\rho\rangle$, for some monic proper polynomial $g_0\in C_\rho[z]$ which is irreducible in $C_\rho[z]$ and $g_0\neq z$.*

Proof. (i) → (ii). If P is prime, then $P=f_0Q\langle X;\rho\rangle\cap R\langle X;\rho\rangle$, where $f_0\in\Gamma_Q^0$ and it is irreducible in Γ_Q. Since $X^{-\partial f_0}f_0\in Z$ we easily get $\partial f_0 = ms$, for some $s \geq 1$. Then $g_0 = f_0 b^{-s}$ is a monic proper polynomial in $C_\rho[z]$. The irreducibility of f_0 in Γ_Q implies the irreducibility of g_0 in $C_\rho[z]$. Finally $P = g_0 Q\langle X; \rho\rangle \cap R\langle X; \rho\rangle$.

(ii) → (i). It is easy to revert the arguments.

Corollary 2.10. *There is a one-to-one correspondence between the following.*

(i) *The set of all R-disjoint prime ideals of* $R\langle X; \rho\rangle$.

(ii) *The set of all Q-disjoint prime ideals of* $Q\langle X; \rho\rangle$.

(iii) *The set of all maximal ideals of Z.*

Moreover, this correspondence associates the R-disjoint prime ideal P of $R\langle X; \rho\rangle$, *the Q-disjoint prime ideal* P^* *of* $Q\langle X; \rho\rangle$ *and the maximal ideal M of Z if* $P^* \cap R\langle X; \rho\rangle = P$ *and* $MQ\langle X; \rho\rangle = P^*$.

Proof. If there is no non-zero R-disjoint ideal of $R\langle X; \rho\rangle$, then the same is true of $Q\langle X; \rho\rangle$ and $Z = C_\rho$ is a field. This establish the result in this case. The other case can easily be proved using Lemma 2.6, Theorem 2.8 and Corollary 2.9.

In particular, we have

Corollary 2.11. *Assume that there exists a non-zero R-disjoint ideal of* $R\langle X; \rho\rangle$. *Then there is a one-to-one correspondence between the following.*

(i) *The set of all R-disjoint prime ideals of* $R\langle X; \rho\rangle$.

(ii) *The set of all prime ideals of* $C_\rho[t]$ *which are different of* $tC_\rho[t]$, *where t is an indeterminate.*

Remark 2.12. Using the results on closed ideals we can also give a one-to-one correspondence between the set of all closed ideals of $R\langle X; \rho\rangle$, the set of all closed ideals of $Q\langle X; \rho\rangle$ and the set of all the ideals of Z, as in Corollary 2.10. It follows that an intersection of closed (prime) ideals of $R\langle X; \rho\rangle$ is non-zero if and only if it is a finite intersection.

3. Prime ideals of $R[X; \rho]$. It is quite easy to describe the prime ideals of $R[X; \rho]$, based on the results of the former section.

Firstly, let I be an ideal of $R[X; \rho]$. We say that X is regular modulo I if the following condition holds: $Xf \in I$ implies $f \in I$ and $gX \in I$ implies $g \in I$, for any f, g in $R[X; \rho]$. It is easy to see that if P is a prime ideal of $R[X;\rho]$ with $X \notin P$, then X is regular modulo P.

We begin this section with the following.

Lemma 3.1. *There is a one-to-one correspondence via contraction between the following.*

(i) *The set of all R-disjoint ideals of $R\langle X;\rho\rangle$.*

(ii) *The set of all R-disjoint ideals I of $R[X;\rho]$ such that X is regular modulo I.*

Proof. If I is an R-disjoint ideal of $R\langle X;\rho\rangle$, then $I_0 = I \cap R[X;\rho]$ is an R-disjoint ideal of $R[X;\rho]$ and X is regular modulo I_0. On the other hand, if J is an R-disjoint ideal of $R[X;\rho]$ such that X is regular modulo J we put $(J) = \sum_{i\geq 0} X^{-i}J$. Then (J) is an ideal of $R\langle X;\rho\rangle$ such that $(J) \cap R[X;\rho] = J$. The rest is clear.

If P is a prime ideal of $R[X;\rho]$, then either $X \in P$ and $P = (P \cap R) + XR[X;\rho]$ or X is regular modulo P and $P \cap R$ is a strongly ρ-prime ideal of R ([1], Lemma 1.3). Since the prime ideals of the first type are determined by the prime ideals of R, we are interested in the prime ideals P with $X \notin P$. In this case, by factoring out $P \cap R$ we may asume $P \cap R = 0$ and R is strongly ρ-prime. We recall the following.

Lemma 3.2 (*c.f.* [1], *Proposition* 1.6). *Let P be an R-disjoint ideal of $R[X;\rho]$ with $X \notin P$. Then P is prime if and only if R is strongly ρ-prime and P is maximal with respect to $P \cap R = 0$.*

As an immediate consequence of our former results we have the following corollaries.

Corollary 3.3. *Let R be a strongly ρ-prime ring. Then there is a one-to-one correspondence via contraction between the following.*

(i) *The set of all R-disjoint prime ideals of $R\langle X;\rho\rangle$.*

(ii) *The set of all R-disjoint prime ideals P of $R[X;\rho]$ with $X \notin P$.*

Corollary 3.4. *Let R be a strongly ρ-prime ring and let P be a nonzero R-disjoint ideal of $R[X;\rho]$. Then P is prime if and only if one of the following conditions is fulfilled.*

(i) *R is prime and $P = XR[X;\rho]$.*

(ii) *$P = f_0Q[X;\rho] \cap R[X;\rho]$, where f_0 is a monic irreducible poly-*

nomial in $C_\rho[z]$ *which is different of* z ($z = X^m b^{-1}$ *as above*).

Remark 3.5. We also have a one-to-one correspondence between the set of all R-disjoint prime ideals of $R[X;\rho]$, the set of all Q-disjoint prime ideals of $Q[X;\rho]$ and the set of all maximal ideals of $C_\rho[z]$, when $Z(Q[X;\rho]) \neq C_\rho$.

On the other hand, it is also possible to define a closure operator in the set of R-disjoint ideals of $R[X;\rho]$ so that the prime ideals become closed. But we do not see any good reason to study this notion.

4. **Nonsingular prime ideals.** In this section we denote by $S(R)$ the (right) singular ideal of R([7], p.30). We recall that a prime ideal P of R is said to be (right) nonsingular if $S(R/P) = 0$. From the results in ([3], § 4) it follows that every prime ideal of the polynomial ring $R[X]$ is nonsingular if and only if every prime ideal of R is nonsingular.

The purpose of this section is to apply the results in the former sections to get necessary and sufficient conditions for every prime ideal of $R\langle X;\rho\rangle$ ($R[X;\rho]$) to be nonsingular. We have the following.

Theorem 4.1. *Every prime ideal of* $R\langle X;\rho\rangle$ *is nonsingular if and only if every* ρ*-prime ideal of* R *is nonsingular.*

Theorem 4.2. *Every prime ideal of* $R[X;\rho]$ *is nonsingular if and only if every prime ideal and every strongly* ρ*-prime ideal of* R *are nonsingular.*

The proof of Theorem 4.1 is a trivial consequence of the next lemmas. Theorem 4.2 can be shown similarly.

We denote by $r_R(a)$ the right annihilator of a in R. Also, if I is a right ideal of R, $I\langle X\rangle$ denotes the right ideal of $R\langle X;\rho\rangle$ whose elements can be written in the form $\sum_{i=-n}^{n} b_i X^i$, $b_i \in I$. Finally we put $T = R\langle X;\rho\rangle$.

Lemma 4.3. $S(T) = S(R)\langle X;\rho\rangle$.

Proof. Suppose that $a \in S(R)$ and let I be a non-zero right ideal of T. If $I \cap R \neq 0$, then it is clear that there exists $0 \neq b \in I \cap R$ such that $ab = 0$. Assume $I \cap R = 0$ and suppose that $ag \neq 0$ for every non-zero polynomial $g \in I$. Hence there exists a non-zero polynomial $f \in I$ such that $\partial(af)$ is of minimal degree s, say, $af = X^s \rho^s(a) a_s + \cdots + aa_0$. Since $a_s R \neq$

0 and $\rho^s(a) \in S(R)$, there exists $r \in R$ with $a_s r \neq 0$ and $\rho^s(a)a_s r = 0$. Thus $0 \neq fr \in I$ and $\partial(afr) < s$, a contradiction. Therefore $r_T(a) \cap I \neq 0$ and so $a \in S(T)$.

Now, let $f = \sum_{i=t}^{n} X^i b_i \in S(T)$, $b_n \neq 0$ $(t \leq n)$. If I is a non-zero right ideal of R, then there exists $0 \neq h \in I\langle X\rangle$ with $fh = 0$. It follows that $r_R(b_n) \cap I \neq 0$. Hence $b_n \in S(R)$ and so $X^n b_n \in S(T)$. Thus $f - X^n b_n \in S(T)$ and repeating this argument we get $f \in S(R)\langle X; \rho\rangle$. This completes the proof.

Lemma 4.4. *Assume that R is ρ-prime and $S(R) = 0$. Then every prime ideal P of T such that $P \cap R = 0$ is nonsingular.*

Proof. If $P = 0$, then P is nonsingular by Lemma 4.3. Thus we may assume $P \neq 0$. Let f_P be the canonical polynomial of P. We have $P = f_P Q\langle X; \rho\rangle \cap R\langle X; \rho\rangle$. If $S(T/P) = I/P \neq 0$, then $I \supsetneqq P$ and so $I \cap R \neq 0$. Take $0 \neq a \in I \cap R$ and suppose J is a non-zero right ideal of R. Since $0 \neq (J\langle X\rangle + P)/P$ and $r_T(a+P)$ is an essential right ideal of T/P, there exists $0 \neq f \in J\langle X\rangle$ such that $af \in P$. We may assume that f is a polynomial. If $\partial f < \mathrm{Min}(P) = n$, then we get $aa_i = 0$ for every left coefficient a_i of f. If $\partial f \geq n$ we write $f = hf_P + r$, where h, $r \in Q\langle X\rangle$ and r is a polynomial with either $r = 0$ or $\partial r < n$. Using the fact that f_P is monic and $f \in JQ\langle X\rangle$ we easily get $r \in JQ\langle X; \rho\rangle$. Also $ar = af - ahf_P \in Q\langle X; \rho\rangle f_P$ and so $ar = 0$. Choose a non-zero ρ-invariant ideal L of R such that $rL \subseteq J\langle X\rangle$. Then $arb = 0$ for some $0 \neq rb \in J\langle X\rangle$. It follows that $r_R(a) \cap J \neq 0$ and therefore $a \in S(R) = 0$, a contradiction.

Lemma 4.5. *If P is a prime nonsingular ideal of $R\langle X; \rho\rangle$ with $P \cap R = 0$, then R is nonsingular.*

Proof. If $P = 0$, then R is nonsingular by Lemma 4.3. Assume $P \neq 0$ and suppose $r_R(a)$ is an essential right ideal of R for some $a \in R$. Let $J/P \neq 0$ a right ideal of T/P. If g and f are proper polynomials of minimal degree m and n in J and P, respectively, then $0 \leq m \leq n$. Assume $m = n$. Therefore $gr\rho^i(a) - \rho^{-n}(br)\rho^i(f) \in J$, for every $r \in R$, where $a = \ell c(f)$ and $b = \ell c(g)$. Hence $grX^i a \in P$, for every $r \in R$, $i \in \mathbb{Z}$, and it follows that $g \in P$. A standard argument shows that $J = P$.

Thus we may assume $m < n$. If $ag \neq 0$ for every $g \in J$ such that $\partial g = m$, then there exists $h \in J$, $\partial h = m$, such that $ah \neq 0$ and $\partial(ah)$ is minimal. We get a contradiction as in the proof of Lemma 4.3. Consequently there exists $0 \neq g \in J$ with $\partial g = m$ and $ag = 0$. This gives $(a+P)(g+P) = \bar{0}$ in T/P, where $0 \neq g+P \in J/P$. Therefore $a+P \in S(T/P) = 0$ and so $a \in P \cap R = 0$. The proof is complete.

Remark 4.6. All this paper was devoted to consider right questions. There are, of course, similar results for the left ρ-quotient ring of R and left nonsingular prime ideals.

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E. CISNEROS AND M. I. GONZÁLEZ
DEPARTAMENTO DE MATEMÁTICAS
UNIVERSIDAD NACIONAL DE ROSARIO
AVDA. PELLEGRINI, 250.
2.000 — ROSARIO-ARGENTINA

M. FERRERO
INSTITUTO DE MATEMÁTICA
UNIVERSIDADE FEDERAL DO RIO
GRANDE DO SUL
90.049 — PORTO ALEGRE-BRASIL

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