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AN ANTI-HOMOMORPHISM FOR THE BRAUER-LONG GROUP

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In [2], F. W. Long constructed a Brauer group $BD(R, H)$ of dimodule algebras for a commutative ring R and a commutative and cocommutative, finitely generated projective Hopf algebra H over R. In this paper we will discuss the following two questions: (1) Given any H-Azumaya algebra A, is the (usual) opposite algebra A° also H-Azumaya? (2) Since H is finitely generated projective, H^* is also a Hopf algebra and $H \cong H^{**}$ as a Hopf algebra. Is there any relation between $BD(R, H)$ and $BD(R, H^*)$?

The answer to (1) is no in general: a counter-example is given below. But there is a natural way to give A° an H^*-dimodule algebra structure and with this structure, $\overline{A^\circ}$ is an H^*-Azumaya R-algebra. Furthermore, the correspondence $[A] \to [\overline{A^\circ}]$ defines an isomorphism between the groups $BD(R, H)$ and $BD(R, H^*)$ giving the answer to (2).

We first recall some definitions.

1. Preliminaries (For more details, look at [2]). Throughout this paper, R is a fixed commutative ring with identity, each $\otimes$ is taken over R and each map is R-linear unless otherwise stated. Moreover H is a commutative and cocommutative, finitely generated projective Hopf algebra over R, ε and Δ denote the counit and diagonalization of H, and the action of Δ is denoted by $\Delta(h) = \sum_{(h)} h_{(1)} \otimes h_{(2)}$.

An R-algebra A is called an *H-module algebra* if A is an H-module such that the H-action map $\rightharpoondown_A: H \otimes A \to A$ is an R-algebra map, that is, for $h \in H$, $a, b \in A$, $h \rightharpoondown (ab) = \sum_{(h)} (h_{(1)} \rightharpoondown a)(h_{(2)} \rightharpoondown b)$ and $h \rightharpoondown 1 = \varepsilon(h)1$.

Similarly an R-algebra A is called an *H-comodule algebra* if A is an H-comodule via $\chi_A: A \to A \otimes H$ such that χ_A is an R-algebra map, that is, for a, b in A, $\chi_A(ab) = \sum_{(a),(b)} a_{(0)} b_{(0)} \otimes a_{(1)} b_{(1)}$ and $\chi_A(1) = 1_A \otimes 1_H$, where $\chi_A(a) = \sum_{(h)} a_{(0)} \otimes a_{(1)}$.

An R-algebra A is called an *H-dimodule algebra* if A is an H-module algebra and an H-comodule algebra such that the following diagram commutes:

$$\begin{array}{ccc} H\otimes_R A & \xrightarrow{\rightharpoonup_A} & A \\ {\scriptstyle I\otimes\chi_A}\downarrow & & \downarrow{\scriptstyle \chi_A} \\ H\otimes_R A\otimes_R H & \xrightarrow[\rightharpoonup_A\otimes I]{} & A\otimes_R H \end{array}$$

Let A be an H-dimodule algebra. The *H-opposite* $\bar{A}$ of A is an isomorphic copy of A as an H-dimodule and the multiplication on $\bar{A}$ is defined by $\bar{a}\bar{b} = \overline{\sum_{(a)}(a_{(1)} \rightharpoonup b)a_{(0)}}$. Let B be another H-dimodule algebra. Then $A \# B$ is an isomorphic copy of $A \otimes_R B$ as H-dimodule and the multiplication on $A \# B$ is defined by $(a \# b)(c \# d) = \sum_{(b)} a(b_{(1)} \rightharpoonup c) \# b_{(0)}d$. The algebras $\bar{A}$ and $A \# B$ are H-dimodule algebras.

Let A be an H-dimodule algebra. We define F_A: $A \# \bar{A} \to \mathrm{End}_R(A)$ and G_A: $\bar{A} \# A \to \mathrm{End}_R(A)^\circ$ by $F_A(a \# \bar{b})(c) = \sum_{(b)} a(b_{(1)} \rightharpoonup c)b_{(0)}$ and $G_A(\bar{a} \# b)(c) = \sum_{(b)}(c_{(1)} \rightharpoonup a)c_{(0)}b$. Both of these maps are homomorphisms of H-dimodule algebras. A is said to be *H-Azumaya* if A is an H-dimodule algebra which is an R-progenerator such that the maps F_A and G_A are isomorphisms of H-dimodule algebras.

Let A, B be H-Azumaya algebras. We say A and B are *H-Brauer equivalent* (denoted by $A \sim_H B$) if there exist H-dimodules M, N which are R-progenerators such that $A \# \mathrm{End}_R(M) \cong B \# \mathrm{End}_R(N)$ as H-dimodule algebras. $\sim_H$ is an equivalence relation which respects the operation $\#$. The quotient set is a group under the multiplication induced by $\#$, with inverse induced by $\bar{\ }$. We denote this group by $BD(R, H)$ and call it the *Brauer group* of *H-dimodule algebras*.

We begin by giving an example for (1).

2. Example. Let us recall first some notations and results from Orzech ([3]). Consider the following data: a commutative ring R, a finite abelian group G, a 2-cocycle f: $G\times G \to U(R)$ of G in the units of R with G acting trivially on $U(R)$ and a bilinear map ϕ from $G\times G$ to $U(R)$. Then $H = RG$ is a Hopf algebra and the H-dimodule algebras are just the G-dimodule R-algebras of ([1], [3]).

Let $A = RG^\phi_f$ be the H-dimodule algebra defined as follows: as an R-module A is freely generated by elements x_σ, $\sigma \in G$, the multiplication is defined by $x_\sigma x_\tau = f(\sigma, \tau)x_{\sigma\tau}$, the G-grading by $\deg_G(x_\sigma) = \sigma$ and the G-action by ${}^\sigma(x_\tau) = \phi(\sigma, \tau)x_\tau$. The H-dimodule algebra A is now an H-Azumaya R-algebra if and only if each of the following two matrices is invertible:

$$(\phi(\alpha,\beta)c_{\alpha,\beta}),\ (\phi(\beta,\alpha^{-1})c_{\alpha,\beta}),\ c_{\alpha,\beta}=f(\alpha^{-1},\beta)f(\alpha^{-1}\beta,\alpha).$$

Let $G=C_3\times C_3\times C_3=\langle\sigma_1\rangle\times\langle\sigma_2\rangle\times\langle\sigma_3\rangle$ and $R=\mathbb{C}$. Suppose that ϕ and f satisfy the tables below $\left(w=\exp\left(\frac{2\pi i}{3}\right)\right)$:

ϕ	σ_1	σ_2	σ_3
σ_1	1	w	1
σ_2	1	1	w
σ_3	w	1	1

f	σ_1	σ_2	σ_3
σ_1	1	w	1
σ_2	w^2	1	w
σ_3	1	w^2	1

We shall now prove that the matrices

$$(\phi(\alpha,\beta)c_{\alpha,\beta}),\ (\phi(\beta,\alpha^{-1})c_{\alpha,\beta}),\ c_{\alpha,\beta}=f(\alpha^{-1},\beta)f(\alpha^{-1}\beta,\alpha)$$

are invertible: for $\alpha_1,\alpha_2,\alpha_3,\beta_1,\beta_2,\beta_3\in\mathbb{Z}$ one may proof

$$\begin{aligned}&f(\sigma_1^{\alpha_1}\sigma_2^{\alpha_2}\sigma_3^{\alpha_3},\sigma_1^{\beta_1}\sigma_2^{\beta_2}\sigma_3^{\beta_3})f(\sigma_2,\sigma_1)^{\beta_2\alpha_1-\beta_1\alpha_2}\cdot\\ &\quad f(\sigma_3,\sigma_1)^{\beta_3\alpha_1-\beta_1\alpha_3}f(\sigma_3,\sigma_2)^{\beta_3\alpha_2-\beta_2\alpha_3}\\ &=f(\sigma_1^{\beta_1}\sigma_2^{\beta_2}\sigma_3^{\beta_3},\sigma_1^{\alpha_1}\sigma_2^{\alpha_2}\sigma_3^{\alpha_3})f(\sigma_1,\alpha_2)^{\beta_2\alpha_1-\beta_1\alpha_2}\cdot\\ &\quad f(\sigma_1,\sigma_3)^{\beta_3\alpha_1-\beta_1\alpha_3}f(\sigma_2,\sigma_3)^{\beta_3\alpha_2-\beta_2\alpha_3}.\end{aligned}$$

Let $\alpha=\sigma_1^{\alpha_1}\sigma_2^{\alpha_2}\sigma_3^{\alpha_3}$ and $\beta=\sigma_1^{\beta_1}\sigma_2^{\beta_2}\sigma_3^{\beta_3}$. Then:

$$\begin{aligned}c_{\alpha,\beta}&=f(\alpha^{-1},\beta)f(\alpha^{-1}\beta,\alpha)\\ &=(w^{-1})^{\beta_1\alpha_2-\beta_2\alpha_1}(1)^{\beta_1\alpha_3-\beta_3\alpha_1}(w^{-1})^{\beta_2\alpha_3-\beta_3\alpha_2}f(\beta,\alpha^{-1})f(\alpha^{-1}\beta,\alpha)\\ &=w^{\beta_2\alpha_1-\beta_1\alpha_2+\beta_3\alpha_2-\beta_2\alpha_3}f(\beta,1)f(\alpha^{-1},\alpha)\\ &=w^{\beta_2\alpha_1-\beta_1\alpha_2+\beta_3\alpha_2-\beta_2\alpha_3}f(\alpha^{-1},\alpha)\end{aligned}$$

and

$$\phi(\alpha,\beta)=\prod_{i,j=1}^{3}\phi(\sigma_i,\sigma_j)^{\beta_j\alpha_i}=w^{\beta_1\alpha_3+\beta_2\alpha_1+\beta_3\alpha_2}.$$

So

$$(\phi(\alpha,\beta)c_{\alpha,\beta})=(w^{\beta_1\alpha_3+\beta_2\alpha_1+\beta_3\alpha_2+\beta_2\alpha_1-\beta_1\alpha_2+\beta_3\alpha_2-\beta_2\alpha_3}f(\alpha^{-1},\alpha)).$$

This matrix is invertible if and only if the matrix $(w^{\beta_1\alpha_3+\beta_2\alpha_1+\beta_3\alpha_2+\beta_2\alpha_1-\beta_1\alpha_2+\beta_3\alpha_2-\beta_2\alpha_3})$ is invertible and this is equivalent to ϕ_1' being non-degenerate where ϕ_1' is the bilinear map defined by

$$\begin{aligned}\phi_1'(\sigma_1^{\alpha_1}\sigma_2^{\alpha_2}\sigma_3^{\alpha_3},\sigma_1^{\beta_1}\sigma_2^{\beta_2}\sigma_3^{\beta_3})&=w^{\beta_1\alpha_3+\beta_2\alpha_1+\beta_3\alpha_2+\beta_2\alpha_1-\beta_1\alpha_2+\beta_3\alpha_2-\beta_2\alpha_3}\\ &=w^{\beta_1(\alpha_3-\alpha_2)+\beta_2(2\alpha_1-\alpha_3)+\beta_3(2\alpha_2)}\end{aligned}$$

$$= w^{\alpha_1(2\beta_2)+\alpha_2(2\beta_3-\beta_1)+\alpha_3(\beta_1-\beta_2)},$$

cf. Proposition 2.8. in [3], and it is easy to see that this is true.

Analogously, the matrix $(\phi(\beta, \alpha^{-1})c_{\alpha,\beta})$ being invertible is equivalent to the bilinear map ϕ_2' being non-degenerate where ϕ_2' is the bilinear map defined by

$$\phi_2'(\sigma_1^{\alpha_1}\sigma_2^{\alpha_2}\sigma_3^{\alpha_3}, \sigma_1^{\beta_1}\sigma_2^{\beta_2}\sigma_3^{\beta_3}) = w^{-\beta_1(2\alpha_2)+\beta_2(\alpha_1-2\alpha_3)+\beta_3(\alpha_2-\alpha_1)}$$
$$= w^{\alpha_1(\beta_2-\beta_3)+\alpha_2(\beta_3-2\beta_1)-\alpha_3(2\beta_2)},$$

and this is also true.

We shall now prove that the matrices $(\phi(\alpha, \beta)d_{\alpha,\beta})$ and $(\phi(\beta, \alpha^{-1})d_{\alpha,\beta})$ with $d_{\alpha,\beta} = f(\beta, \alpha^{-1})f(\alpha, \alpha^{-1}\beta)$ are not invertible: the matrix $(\phi(\alpha, \beta)d_{\alpha,\beta})$ being invertible is equivalent to the bilinear map ϕ_3' being non-degenerate where ϕ_3' is the bilinear map defined by

$$\phi_3'(\sigma_1^{\alpha_1}\sigma_2^{\alpha_2}\sigma_3^{\alpha_3}, \sigma_1^{\beta_1}\sigma_2^{\beta_2}\sigma_3^{\beta_3}) = w^{\beta_1(\alpha_2+\alpha_3)+\beta_2\alpha_3} = w^{\alpha_2\beta_1+\alpha_3(\beta_1+\beta_2)}$$

and the matrix $(\phi(\beta, \alpha^{-1})d_{\alpha,\beta})$ being invertible is equivalent to the bilinear map ϕ_4' being non-degenerate where ϕ_4' is the bilinear map defined by

$$\phi_4'(\sigma_1^{\alpha_1}\sigma_2^{\alpha_2}\sigma_3^{\alpha_3}, \sigma_1^{\beta_1}\sigma_2^{\beta_2}\sigma_3^{\beta_3}) = w^{-\beta_2\alpha_1-\beta_3(\alpha_1+\alpha_2)} = w^{-\alpha_1(\beta_2+\beta_3)-\alpha_2\beta_3}.$$

It is clear ϕ_3' that ϕ_4' and are both degenerate, so the matrices $(\phi(\alpha, \beta)d_{\alpha,\beta})$ and $(\phi(\beta, \alpha^{-1})d_{\alpha,\beta})$ with $d_{\alpha,\beta} = f(\beta, \alpha^{-1})f(\alpha, \alpha^{-1}\beta)$ are not invertible.

In the case we consider, $H^* = RG^* = \langle\chi_1\rangle \times \langle\chi_2\rangle \times \langle\chi_3\rangle \cong RG$ with χ_1, χ_2 and χ_3 defined by $\langle\chi_i, \sigma_j\rangle \dot{=} w^{\sigma_{ij}}$ for $i, j = 1, 2, 3$. We can give A an H^*-dimodule algebra structure as follows: for $x \in A$ and $\chi \in G^*$, $\deg_{G^*}(x) = \chi$ if and only if ${}^{\sigma}x = \chi(\sigma)x$ for all $\sigma \in G$ and ${}^{\chi}x = \chi(\deg_G(x))x$. It is easy to see that for this structure $A = RG^{*\psi}_{g}$ where ψ and g satisfy the tables below:

ψ	χ_1	χ_2	χ_3
χ_1	1	1	w
χ_2	w	1	1
χ_3	1	w	1

g	χ_1	χ_2	χ_3
χ_1	1	w	w^2
χ_2	w^2	1	1
χ_3	w	1	1

We shall now check if the matrices

$$(\psi(\alpha, \beta)c'_{\alpha,\beta}), (\psi(\beta, \alpha^{-1})c'_{\alpha,\beta}), c'_{\alpha,\beta} = g(\alpha^{-1}, \beta)g(\alpha^{-1}\beta, \alpha)$$

and

$$(\psi(\alpha, \beta)d'_{\alpha,\beta}), (\psi(\beta, \alpha^{-1})d'_{\alpha,\beta}), d'_{\alpha,\beta} = g(\beta, \alpha^{-1})g(\alpha, \alpha^{-1}\beta)$$

are invertible. The matrix $(\psi(\alpha, \beta)c'_{\alpha,\beta})$ being invertible is equivalent to the bilinear map ψ'_1 being non-degenerate where ψ'_1 is the bilinear map defined by

$$\psi'_1(\chi_1^{\alpha_1}\chi_2^{\alpha_2}\chi_3^{\alpha_3}, \chi_1^{\beta_1}\chi_2^{\beta_2}\chi_3^{\beta_3}) = w^{\beta_1\alpha_3+\beta_2(\alpha_1+\alpha_3)} = w^{\alpha_1\beta_2+\alpha_3(\beta_1+\beta_2)}.$$

The matrix $(\psi(\beta, \alpha^{-1})c'_{\alpha,\beta})$ being invertible is equivalent to the bilinear map ψ'_2 being non-degenerate where ψ'_2 is the bilinear map defined by

$$\psi'_2(\chi_1^{\alpha_1}\chi_2^{\alpha_2}\chi_3^{\alpha_3}, \chi_1^{\beta_1}\chi_2^{\beta_2}\chi_3^{\beta_3}) = w^{-\beta_1\alpha_2-\beta_3(\alpha_1+\alpha_2)} = w^{-\alpha_1\beta_3-\alpha_2(\beta_1+\beta_3)}.$$

It is clear that ψ'_1 and ψ'_2 are both degenerate, so the matrices

$$(\psi(\alpha, \beta)c'_{\alpha,\beta}), (\psi(\beta, \alpha^{-1})c'_{\alpha,\beta}), c'_{\alpha,\beta} = g(\alpha^{-1}, \beta)g(\alpha^{-1}\beta, \alpha)$$

are not invertible. The matrix $(\psi(\alpha, \beta)d'_{\alpha,\beta})$ being invertible is equivalent to the bilinear map ψ'_3 being non-degenerate where ψ'_3 is the bilinear map defined by

$$\begin{aligned}\psi'_3(\chi_1^{\alpha_1}\chi_2^{\alpha_2}\chi_3^{\alpha_3}, \chi_1^{\beta_1}\chi_2^{\beta_2}\chi_3^{\beta_3}) &= w^{\beta_1(2\alpha_2-\alpha_3)+\beta_2(\alpha_3-\alpha_1)+\beta_3(2\alpha_1)} \\ &= w^{\alpha_1(2\beta_3-\beta_2)+\alpha_2(2\beta_1)+\alpha_3(\beta_2-\beta_1)}.\end{aligned}$$

The matrix $(\psi(\beta, \alpha^{-1})d_{\alpha,\beta})$ being invertible is equivalent to the bilinear map ψ'_4 being non-degenerate where ψ'_4 is the bilinear map defined by

$$\begin{aligned}\psi'_4(\chi_1^{\alpha_1}\chi_2^{\alpha_2}\chi_3^{\alpha_3}, \chi_1^{\beta_1}\chi_2^{\beta_2}\chi_3^{\beta_3}) &= w^{\beta_1(\alpha_2-2\alpha_3)-\beta_2(2\alpha_1)+\beta_3(\alpha_1-\alpha_2)} \\ &= w^{\alpha_1(\beta_3-2\beta_2)+\alpha_2(\beta_1-\beta_3)-\alpha_3(2\beta_1)}.\end{aligned}$$

It is clear that ψ'_3 and ψ'_4 are both non-degenerate, so the matrices $(\psi(\alpha, \beta)d'_{\alpha,\beta})$ and $(\psi(\beta, \alpha^{-1})d'_{\alpha,\beta})$ with $d'_{\alpha,\beta} = g(\beta, \alpha^{-1})g(\alpha, \alpha^{-1}\beta)$ are invertible.

Note that if you relabel the σ's by letting $\tau_1 = \sigma_2$, $\tau_2 = \sigma_1$, $\tau_3 = \sigma_3$, and write down the generating tables for ϕ and f°, then these are identical to the generating tables for ψ and g as is us pointed out by Beattie. So, using this, $(\phi(\alpha, \beta)c_{\alpha,\beta})$ and $(\phi(\beta, \alpha^{-1})c_{\alpha,\beta})$ being invertible is equivalent to $(\psi(\alpha, \beta)d'_{\alpha,\beta})$ and $(\psi(\beta, \alpha^{-1})d'_{\alpha,\beta})$ being invertible and $(\phi(\alpha, \beta)d_{\alpha,\beta})$ and $(\phi(\beta, \alpha^{-1})d_{\alpha,\beta})$ being invertible is equivalent to $(\psi(\alpha, \beta)c'_{\alpha,\beta})$ and $(\psi(\beta, \alpha^{-1})c'_{\alpha,\beta})$ being invertible.

So we obtain that A° is an H^*-Azumaya algebra but A is not. The property "an H-Azumaya algebra is not H^*-Azumaya" is not true in general. For example, consider an Azumaya algebra A with trivial H-dimodule structure. Then A is an H-Azumaya algebra as well as an H^*-Azumaya algebra. But that the opposite algebra A° of an H-Azumaya algebra is H^*-Azumaya is always true if H is a commutative and cocommutative, finitely generated projective Hopf algebra. This is what we are going to show now.

3. Proposition. *Let H be a commutative and cocommutative, finitely generated projective Hopf algebra. Then H^* is also a Hopf algebra and $H \cong H^{**}$ as Hopf algebras. Let M be an H-dimodule with structure maps $H \otimes_R M \to M: h \otimes m \to (h \rightharpoondown m)$* and *$M \to M \otimes_R H: m \to \sum_{(m)} m_{(0)} \otimes m_{(1)}$. Then :*

(1) *M is a left H^*-module by $H^* \otimes_R M \to M: f \otimes m \to (f \rightharpoondown m) = \sum_{(m)} m_{(0)} f(m_{(1)})$;*

(2) *M is a right H^*-comodule by $M \to M \otimes_R H^*: m \to \sum_{(m)} m^{(0)} \otimes m^{(1)}$ where $\sum_{(m)} m^{(0)} m^{(1)}(h) = (h \rightharpoondown m)$ for all $h \in H$;*

(3) $\sum_{(f \rightharpoondown m)} (f \rightharpoondown m)^{(0)} \otimes (f \rightharpoondown m)^{(1)} = \sum_{(m)} (f \rightharpoondown m^{(0)}) \otimes m^{(1)}$.

So M is an H^*-dimodule. Furthermore, if A is an H-dimodule R-algebra, then A is an H^*-dimodule R-algebra for the H^*-dimodule structure defined above.

Proof. We refer to Long in [2] for a proof that H^* is also a finitely generated projective Hopf algebra. For a proof of (1) and (2) we refer to Long in [2] and Pareigis in [4].

(3). Using (1) we obtain

$$\sum_{(f \rightharpoondown m)} (f \rightharpoondown m)^{(0)} \otimes (f \rightharpoondown m)^{(1)} = \sum_{(m)(m_{(0)})} (m_{(0)})^{(0)} \otimes (m_{(0)})^{(1)} f(m_{(1)}) \ (*)$$

and

$$\sum_{(m)} (f \rightharpoondown m^{(0)}) \otimes m^{(1)} = \sum_{(m)(m^{(0)})} (m^{(0)})_{(0)} f((m^{(0)})_{(1)}) \otimes m^{(1)} \ (**).$$

Let h be an element of H. We let $(*)$ work on h and we obtain:

$$\begin{aligned} \sum_{(m)(m_{(0)})} (m_{(0)})^{(0)} (m_{(0)})^{(1)}(h) f(m_{(1)}) &= \sum_{(m)} (h \rightharpoondown m_{(0)}) f(m_{(1)}) \\ &= \sum_{(h \rightharpoondown m)} (h \rightharpoondown m)_{(0)} f((h \rightharpoondown m)_{(1)}) \\ &= f \rightharpoondown (h \rightharpoondown m). \end{aligned}$$

Now we let $(**)$ work on h and we obtain:

$$\begin{aligned} \sum_{(m)(m^{(0)})} (m^{(0)})_{(0)} f((m^{(0)})_{(1)}) m^{(1)}(h) &= \sum_{(m)} m^{(1)}(h) (f \rightharpoondown m^{(0)}) \\ &= f \rightharpoondown (\textstyle\sum_{(m)} m^{(1)}(h) m^{(0)}) \\ &= f \rightharpoondown (h \rightharpoondown m) \end{aligned}$$

proving (3).

Let A be an H-dimodule R-algebra. Then we obtain:

(a) $f \rightharpoondown (ab) = \sum_{(ab)} (ab)_{(0)} f((ab)_{(1)}) = \sum_{(a)(b)} a_{(0)} b_{(0)} f(a_{(1)} b_{(1)})$
$$= \sum_{(a)(b)(f)} a_{(0)} b_{(0)} f^{(1)}(a_{(1)}) f^{(2)}(b_{(1)})$$
$$= \sum_{(f)} \sum_{(a)} a_{(0)} f^{(1)}(a_{(1)}) \sum_{(b)} b_{(0)} f^{(2)}(b_{(1)}) = \sum_{(f)} (f^{(1)} \rightharpoondown a)(f^{(2)} \rightharpoondown b)$$

(b) $\sum_{(ab)} (ab)^{(0)} (ab)^{(1)}(h) = (h \rightharpoondown (ab)) = \sum_{(h)} (h^{(1)} \rightharpoondown a)(h^{(2)} \rightharpoondown b)$
$$= \sum_{(h)} \sum_{(a)} a^{(0)} a^{(1)}(h^{(1)}) \sum_{(b)} b^{(0)} b^{(1)}(h^{(2)})$$
$$= \sum_{(a)(b)} a^{(0)} b^{(0)} \sum_{(h)} a^{(1)}(h^{(1)}) b^{(1)}(h^{(2)})$$
$$= \sum_{(a)(b)} a^{(0)} b^{(0)} (a^{(1)} b^{(1)})(h).$$

So in this case A turns out to be an H^*-dimodule R-algebra.

4. Notation. For any H-dimodule M, if we consider the H^*-dimodule structure, we write M_{H^*}. Elements of M_{H^*} are denote by b^* for $b \in M$.

5. Lemma. *Let A be an H-dimodule algebra. Then $\overline{(A_{H^*})}$ is isomorphic to A as H-dimodule. Multiplication on $\overline{(A_{H^*})}$ is given by:*

$$\overline{a^* b^*} = \overline{(\sum_{(b)} b_{(0)} (b_{(1)} \rightharpoondown a))^*}.$$

Proof. For $a, b \in A$, we obtain:

$$\overline{a^* b^*} = \overline{(\sum_{(a)} (a^{(1)} \rightharpoondown b) a^{(0)})^*} = \overline{(\sum_{(a)(b)} b_{(0)} a^{(1)}(b_{(1)}) a^{(0)})^*} = \overline{(\sum_{(b)} b_{(0)} (b_{(1)} \rightharpoondown a))^*}.$$

6. Lemma. *Let A, B be H-dimodule algebras. Then $(A_{H^*}) \# (B_{H^*})$ is isomorphic to $A \otimes_R B$ as H-dimodule. Multiplication on $(A_{H^*}) \# (B_{H^*})$ is given by:*

$$(a^* \# b^*)(c^* \# d^*) = \sum_{(c)} (ac_{(0)})^* \# ((c_{(1)} \rightharpoondown b)d)^*.$$

Proof. For $a, c \in A$ and $b, d \in B$, we obtain:

$$\begin{aligned}(a^* \# b^*)(c^* \# d^*) &= \sum_{(b)} a^*(b^{*(1)} \rightharpoondown c^*) \# b^{*(0)} d^* \\ &= \sum_{(b)(c)} a^* c^*_{(0)} b^{*(1)}(c^*_{(1)}) \# b^{*(0)} d^* \\ &= \sum_{(b)(c)} (ac_{(0)})^* \# (b^{(1)}(c_{(1)}) b^{(0)} d)^* \\ &= \sum_{(c)} (ac_{(0)})^* \# ((c_{(1)} \rightharpoondown b)d)^*.\end{aligned}$$

7. Lemma. *Let A be an H-dimodule algebra. Then the R-linear map f from $(A^\circ)_{H^*} \# \overline{((A^\circ)_{H^*})}$ to $(\bar{A} \# A)^\circ$ defined by $f(a^{\circ *} \# \overline{b^{\circ *}}) = (\bar{b} \# a)^\circ$ is an isomorphism of H^*-dimodule algebras.*

Proof. Let a, b, c, d be elements of A. Then we obtain:

$$
\begin{aligned}
f((a^{\circ *} \# \overline{b^{\circ *}})(c^{\circ *} \# \overline{d^{\circ *}})) &= f(\textstyle\sum_{(c)}(a^{\circ}c_{(0)}{}^{\circ})^{*} \# \overline{(c_{(1)} \neg b)^{\circ *}d^{\circ *}}) \\
&= f(\textstyle\sum_{(c)(d)}(a^{\circ}c_{(0)}{}^{\circ})^{*} \# \overline{(d_{(0)}{}^{\circ}(c_{(1)}d_{(1)} \neg b)^{\circ})^{*}}) \\
&= f(\textstyle\sum_{(c)(d)}(c_{(0)}a)^{\circ *} \# \overline{((c_{(1)}d_{(1)} \neg b)d_{(0)})^{\circ *}}) \\
&= (\textstyle\sum_{(c)(d)}\overline{(c_{(1)}d_{(1)} \neg b)d_{(0)}} \# c_{(0)}a)^{\circ} \\
&= (\textstyle\sum_{(c)}\bar{d}\,\overline{c_{(1)} \neg b} \# c_{(0)}a)^{\circ} \\
&= ((\bar{d} \# c)(\bar{b} \# a))^{\circ} = (\bar{b} \# a)^{\circ}(\bar{d} \# c)^{\circ} \\
&= f(a^{\circ *} \# \overline{b^{\circ *}})f(c^{\circ *} \# \overline{d^{\circ *}}).
\end{aligned}
$$

Since H^* is cocommutative (resp. commutative) it is easy to see that f is an H^*-module (resp. H^*-comodule) homomorphism.

8. Lemma. *The following diagram is commutative:*

$$
\begin{array}{ccc}
(A^{\circ})_{H^*} \# \overline{((A^{\circ})_{H^*})} & \xrightarrow{F_{(A^{\circ})_{H^*}}} & \mathrm{End}_R((A^{\circ})_{H^*}) \\
f\downarrow & & \downarrow Id \\
(\bar{A} \# A)^{\circ} & \xrightarrow[G_{A^{\circ}}]{} & \mathrm{End}_R(A)
\end{array}
$$

Proof. For $a, b, c \in A$, we obtain:

(1) $(G \circ f)(a^{\circ *} \# \overline{b^{\circ *}})(c) = G(\bar{b} \# a)(c) = \sum_{(c)}(c_{(1)} \neg a)c_{(0)}a.$

(2) $F_{(A^{\circ})_{H^*}}(a^{\circ *} \# \overline{b^{\circ *}})(c^{\circ *})$

$$
\begin{aligned}
&= \textstyle\sum_{(b)} a^{\circ *}((b^{\circ *})^{(1)} \neg c^{\circ *})(b^{\circ *})^{(0)} \\
&= (\textstyle\sum_{(b)(c)} a^{\circ}c^{\circ}_{(0)}b^{\circ(1)}(c^{\circ}_{(1)})b^{\circ(0)})^{*} \\
&= (\textstyle\sum_{(c)} a^{\circ}c^{\circ}_{(0)}(c^{\circ}_{(1)} \neg b^{\circ}))^{*} = (\textstyle\sum_{(c)}(c_{(1)} \neg b)c_{(0)}a)^{\circ *}.
\end{aligned}
$$

9. Corollary. *The map G_A is an isomorphism of H-dimodule algebras if and only if $F_{(A^{\circ})_{H^*}}$ is an isomorphism of H^*-dimodule algebras.*

In a similar way, we obtain:

10. Lemma. *Let A be an H-dimodule algebra. Then the R-linear map g from $\overline{((A^{\circ})_{H^*})} \# (A^{\circ})_{H^*}$ to $(A \# \bar{A})^{\circ}$ defined by $g(\overline{a^{\circ *}} \# b^{\circ *}) = (b \# \bar{a})^{\circ}$ is an isomorphism of H^*-dimodule algebras.*

11. Lemma. *The following diagram is commutative:*

$$\begin{array}{ccc}
\overline{((A^{\circ})_{H^*})} \ \# \ (A^{\circ})_{H^*} & \xrightarrow{G_{(A^{\circ})_{H^*}}} & \mathrm{End}_R((A^{\circ})_{H^*})^{\circ} \\
g \downarrow & & \downarrow Id \\
(A \ \# \ \bar{A})^{\circ} & \xrightarrow[F_{A^{\circ}}]{} & \mathrm{End}_R(A)^{\circ}
\end{array}$$

12. Corollary. *The map F_A is an isomorphism of H-dimodule algebras if and only if $G_{(A^{\circ})_{H^*}}$ is an isomorphism of H^*-dimodule algebras.*

So we obtain the following result:

13. Theorem. *Let H be a commutative and cocommutative, finitely generated projective Hopf algebra. Then an H-dimodule algebra A is an H-Azumaya R-algebra if and only if A° is an H^*-Azumaya R-algebra. So there is a one-one correspondence between the H-Azumaya R-algebras and the H^*-Azumaya R-algebras.*

14. Lemma. *Let A, B be H-dimodule R-algebras. Then the R-linear map f from $((A \# B)^{\circ})_{H^*}$ to $(B^{\circ})_{H^*} \# (A^{\circ})_{H^*}$ defined by $f((a \# b)^{\circ*}) = b^{\circ*} \# a^{\circ*}$ is an H^*-dimodule R-algebra isomorphism.*

Proof. For $a, c \in A$ and $b, d \in B$ we obtain:

$$\begin{aligned}
f((a \# b)^{\circ*}(c \# d)^{\circ*}) &= f(((c \# d)(a \# b))^{\circ*}) \\
&= f((\textstyle\sum_{(d)} c(d_{(1)} \rightharpoondown a) \# d_{(0)}b)^{\circ*}) \\
&= \textstyle\sum_{(d)} (d_{(0)}b)^{\circ*} \# (c(d_{(1)} \rightharpoondown a))^{\circ*} \\
&= \textstyle\sum_{(d)} (b^{\circ}d_{(0)}{}^{\circ})^* \# ((d_{(1)} \rightharpoondown a)^{\circ}c^{\circ})^* \\
&= (b^{\circ*} \# a^{\circ*})(d^{\circ*} \# c^{\circ*}) \\
&= f((a \# b)^{\circ*})f((c \# d)^{\circ*}).
\end{aligned}$$

15. Lemma. *Let A be an H-dimodule algebra. Then the R-linear map g from $\overline{((A^{\circ})_{H^*})}$ to $((\bar{A})^{\circ})_{H^*}$ defined by $g(\overline{a^{\circ*}}) = \bar{a}^{\circ*}$ is an H^*-dimodule isomorphism.*

Proof. For $a, b \in A$, we obtain:

$$\begin{aligned}
g(\overline{a^{\circ*}b^{\circ*}}) &= g(\overline{\textstyle\sum_{(b)}(b_{(0)}{}^{\circ}(b_{(1)} \rightharpoondown a)^{\circ})^*}) = g(\overline{\textstyle\sum_{(b)}((b_{(1)} \rightharpoondown a)b_{(0)})^{\circ*}}) \\
&= \textstyle\sum_{(b)}\overline{((b_{(1)} \rightharpoondown a)b_{(0)})}^{\circ*} = (\bar{b}\bar{a})^{\circ*} = (\bar{a})^{\circ*}(\bar{b})^{\circ*} \\
&= g(\overline{a^{\circ*}})g(\overline{b^{\circ*}}).
\end{aligned}$$

We conclude from the foregoing:

16. Theorem. *Let H be a commutative and cocommutative, finitely generated projective Hopf algebra. Then the assignment $[A] \to \overline{[A^{\circ}]}$ defines an isomorphism of groups θ between $BD(R, H)$ and $BD(R, H^*)$.*

17. Remarks. 1. If we restrict θ to $BM(R, H)$ (resp. $BC(R, H)$) the image becomes $BC(R, H^*)$ (resp. $BM(R, H^*)$). Furthermore, if $[A] \in BM(R, H)$ or $[A] \in BC(R, H)$, $\theta([A]) = [A]$.

2. Let $BAz(R, H)$ denote the set of central classes in $BD(R, H)$. This is not always a group as is noted by M. Orzech in [3]. Consider the anti-isomorphism of groups between $BD_g(R, H)$ and $BD_g(R, H^*)$ which maps $[A]$ to $[A^{\circ}]$. If we restrict this anti-isomorphism to $BAz_g(R, H)$, then the image is $BAz_g(R, H^*)$. Using this, we may conclude that $BAz_g(R, H)$ is a group if and only if $BAz_g(R, H^*)$ is a group and that $BAz_g(R, H)$ is the whole of $BD_g(R, H)$ if and only if $BAz_g(R, H^*)$ is the whole of $BD_g(R, H^*)$.

18. Application. Let F be a field, char $F = p > 0$ and $H = F[X]/(X^p - X)$. H is a Hopf algebra via $\Delta(x) = x \otimes 1 + 1 \otimes x$, $\varepsilon(x) = 0$ and $S(x) = -x$. The dual H^* of H is FC_p where C_p is the cyclic group of order p (cf. Proposition 5.1. in [2]). So $BD(F, H) = BD(F, C_p)$.

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