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A NOTE ON ANDERSON-ANDERSON-JOHNSON QUESTIONS

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In this paper a ring means a commutative integral domain. Also dimension means Krull dimension. A ring A is called a locally factorial ring if A_M is a factorial ring (that is, a unique factorization domain) for each maximal ideal M of A. Let $X^{(1)}(A)$ be the set of height one prime ideals of A.

Let A be a Krull ring. [2, §2] asks the following questions :

(Q-1) Let P be a non-zero prime ideal of A. If $P \cap Q = PQ$ for each prime ideal Q of A with $\mathrm{ht}(Q) \leqq 2$ that is incomparable with P, then is P a maximal ideal or an invertible ideal of A ?

(Q-2) Are the following conditions equivalent ?

(1) A is a locally factorial ring with $\dim(A) \leqq 2$.

(2) $P \cap Q = PQ$ for each incomparable prime ideals P, Q of A.

(3) Each ideal of A is a product of primary ideals of A.

Let B be a Noetherian ring with quotient field L, K a finite algebraic extension field of L and A the integral closure of B in K. By Mori-Nagata's Integral Closure Theorem, A is a Krull ring. Such rings are important examples of Krull rings. We will call these rings Krull rings of Mori-Nagata type. The aim of this note is to answer (Q-1) and (Q-2) for Krull rings of Mori-Nagata type. We will prove that the answers for (Q-1) and (Q-2) are 'yes' for Krull rings of Mori-Nagata type.

Lemma 1 (Mori-Nagata's Integral Closure Theorem [5, (33. 10) Theorem]). *Let B be a Noetherian ring, L the quotient field of B, K a finite algebraic extension of L and A the integral closure of B in K. Then*

(1) *A is a Krull ring.*

(2) *Let P be a prime ideal of A and $P' = P \cap B$. Then the quotient field of A/P is a finite algebraic extension of that of B/P'.*

(3) *For each prime ideal P' of B, there exists only a finite number of prime ideals of A lying over P'.*

Lemma 2 (Nagata's Theorem [5, (33. 12) Theorem]). *Let A be a Krull ring of Mori-Nagata type. If $\dim(A) \leqq 2$, then A is a Noetherian ring.*

Lemma 3 (Ratliff [6, Lemma 2. 1]). *Let A be a Noetherian semilocal ring, P a prime ideal of A. Then there exists only a finite number of prime ideals Q of A such that $Q \supset P$, $\mathrm{ht}(Q/P) = 1$ and $\mathrm{ht}(Q) \geqq \mathrm{ht}(P)+2$.*

The proofs of the following Lemma 4 and Lemma 5 appear on [4]. But we will prove them here again for convenience.

Lemma 4 (Goto). *Let A be a Krull ring of Mori-Nagata type, and P a prime ideal of A. Then we have $\bigcap_{i=1}^{\infty} P^i A_P = (0)$.*

Proof. Let B, L and K be as in Lemma 1. Set $P \cap B = P'$ and $B - P' = S$. Then A_S is the integral closure of B_S in K. Thus we may assume that B is a Noetherian local ring and $P \cap B$ is the maximal ideal of B. We may use the induction on $\dim(B)$. Set $\dim(B) = d$. If $d = 0, 1$, then the statement is clear. Suppose that $d \geqq 2$ and the statement holds for all positive integers lower than d. Set $[K : L] = n$. Then $K = \sum_{i=1}^{n} Lx_i$ for some elements $x_1, x_2, \ldots, x_n$ of A. Hence we may assume that $K = L$. Also we may assume that $\mathrm{ht}(P) \geqq 2$. Set $\{Q \in X^{(1)}(A) ; Q \subset P\} = \sum$. Let $Q \in \sum$ and set $Q \cap B = Q'$. Then $Q' \neq (0)$, $B/Q' \subset A/Q$ and the quotient field of A/Q is a finite algebraic extension of that of B/Q' by Lemma 1(2). Let R be the integral closure of B/Q' in the quotient field of A/Q. We have $\dim(B/Q') \leqq d-1$. Choose a prime ideal N of R such that $N \cap (A/Q) = P/Q$. Since $\dim(R) \leqq d-1$, we have

$$(0) = \bigcap_{i=1}^{\infty} N^i R_N \supset \bigcap_{i=1}^{\infty} P^i (A_P/QA_P).$$

Thus $\bigcap_{i=1}^{\infty} P^i A_P \subset \bigcap_{i=1}^{\infty} QA_P$. Since $\dim(A_P) \geqq 2$, it follows that $\bigcap_{Q \in \Sigma} QA_P = (0)$. Thus $\bigcap_{i=1}^{\infty} P^i A_P = (0)$.

Lemma 5 ([4, (4. 2)]). *Let A be a Krull ring of Mori-Nagata type, and $P \in X^{(1)}(A)$. If $P \cap Q = PQ$ for each $Q \in X^{(1)}(A)$ such that $Q \neq P$, then P is an invertible ideal of A.*

Proof. Let M be a maximal ideal of A containing P. Set $A_M = A'$, $MA_M = M'$ and $PA_M = P'$. It suffices to show that P' is a principal ideal of A' (cf. [1, Corollary 2. 2]). If $P' = P'M'$, then

$$P' = P'M' = P'M'^2 = P'M'^3 = \cdots = (0)$$

by Lemma 4 ; a contradiction. Therefore $P' \supsetneqq P'M'$. Since A' is a Krull ring, A' is an atomic ring, that is, each element of A' is a product of irreducible elements of A'. Choose $a' \in P' - P'M'$. Then a' is an irreducible element of A'. Because, if a' is a reducible element with $a' =$

$p'_1 p'_2 \cdots p'_n$, where $n \geqq 2$ and each p'_i is irreducible element, then $p'_i \in P'$ for some i. Thus $a' \in P'M'$; a contradiction. Let $Q' \in X^{(1)}(A')$ which is distinct with P'. If $a' \in Q'$, then $a' \in P' \cap Q' = P'Q' \subset P'M'$; a contradiction. Thus P' is the only height one prime ideal containing a'. Let v be the valuation of the quotient field of A' with the valuation ring $A'_{P'}$. Then we have $v(a') = v(b')$ for each $b' \in P' - P'M'$. Because if, say, $v(a') \lneqq v(b')$, then there exists x' of M' such that $x' = b'/a'$. Then $b' = a'x' \in P'M'$; a contradiction. Thus $P' = a'A' \cup P'M'$. It follows that $P' = a'A'$, namely a principal ideal of A'.

The idea of the proof of the following Theorem 6 is similar with that of [2, Theorem 2. 1].

Theorem 6. *Let A be a Krull ring of Mori-Nagata type. Let P be a non-zero prime ideal of A. If $P \cap Q = PQ$ for each prime ideal Q of A with* $\mathrm{ht}(Q) \leqq 2$ *that is incomparable with P, then P is a maximal ideal or an invertible ideal of A.*

Proof. Let B, L and K be as in Lemma 1. Choose a maximal ideal M of A which contains P. Set $B \cap M = M'$. By Lemma 1(3) only a finite number of maximal ideals $M_1, M_2, \ldots, M_n$ of A lie over M'. We set $M = M_1$. Choose $s_2 \in M_2 - M, \ldots, s_n \in M_n - M$. Set $B[s_2, \ldots, s_n] = B_0$. Then M is the only maximal ideal of A lying over $B_0 \cap M$. Therefore we may assume that M is the only maximal ideal of A lying over M'. Set $B - M' = S$. Then A_S is the integral closure of B_S in K. We have $P_S \cap Q = P_S Q$ for each prime ideal Q of A_S with $\mathrm{ht}(Q) \leqq 2$ and incomparable with P_S. Moreover M_S is the only maximal ideal of A_S lying over M'_S. If P_S is an invertible ideal of A_S, then P is an invertible ideal of A by Lemma 5 and if P_S is a maximal ideal of A_S, then P is a maximal ideal of A. Therefore we may assume that A is a quasi-local ring with the maximal ideal M, B is a Noetherian local ring with the maximal ideal M' and $M \cap B = M'$. We assume that P is neither maximal nor invertible. We will derive a contradiction. We have $P \supsetneqq PM$ by Lemma 4. Choose $x \in P - PM$. Then $B[x]$ is a Noetherian local ring. We set $M \cap B[x] = M''$, $P \cap B[x] = P''$ and $P'' \cap B = P'$. Choose a prime ideal Q'' of $B[x]$ which is minimal over $xB[x]$ and contained in P''. We have $\mathrm{ht}(Q'') = 1$. Choose a prime ideal Q of A lying over Q''. Then we have $\mathrm{ht}(Q) = 1$. If P and Q are incomparable, then $x \in P \cap Q = PQ \subset PM$; a contradiction. Therefore P and Q are comparable. Since $\mathrm{ht}(P) \geqq 2$, we have $P \supset Q$. By Lemma 3, there exists only a finite number of prime ideals N'' of $B[x]$ such that $\mathrm{ht}(N''/Q'') = 1$ and $\mathrm{ht}(N'') \geqq 3$. Denote them by $N''_1, \ldots, N''_t$. We have $M'' \not\subset P'' \cup N''_1 \cup \cdots \cup N''_t$. Choose an element y of $B[x]$ such that $y \in M''$ and $y \notin P'' \cup N''_1 \cup \cdots \cup N''_t$. Choose a prime ideal Q''_0/Q'' of $B[x]/Q''$ which is minimal over the element $y+Q''$. Then $y \in Q''_0$. We have $\mathrm{ht}(Q''_0/Q'') = 1$. Since $y \notin N''_i$. we see that $Q''_0 \neq N''_i$ $(1 \leqq i \leqq t)$. Therefore $\mathrm{ht}(Q''_0) = 2$. Choose a prime ideal Q_0 of A containing Q and lying over Q''_0. We have $\mathrm{ht}(Q_0) = 2$. Since $y \notin P''$, also we have $Q_0 \not\subset P$. If P and Q_0 are comparable, then $\mathrm{ht}(P) \leqq \mathrm{ht}(Q_0) - 1$ and hence $\mathrm{ht}(P) = 1$; a contradiction. Therefore P and Q_0 are incomparable. Hence we have $x \in P \cap Q_0 = PQ_0 \subset PM$; which is a contradiction.

Remark 7 ([4, (3. 5)]). Let A be a Krull ring. Suppose that (Q−1) is 'yes' for A. Let $P \in X^{(1)}(A)$. If $P \cap Q = PQ$ for each $Q \in X^{(1)}(A)$ such that $Q \neq P$, then P is an invertible ideal of A.

Theorem 8. *Let A be a Krull ring of Mori-Nagata type. Then the following conditions are equivalent for A* :

(1) *A is a locally factorial ring with* $\dim(A) \leqq 2$.

(2) $P \cap Q = PQ$ *for each incomparable prime ideals P, Q of A.*

(3) *Each ideal of A is a product of primary ideals of A.*

Proof. (3) implies (1) by [3, Corollary 6]. It is straightforward to see that (1) implies (2). Assume (2). We must prove (3). By Theorem 6 we have $\dim(A) \leqq 2$. Then Lemma 2 implies that A is a Noetherian ring. By [2, Corollary (2. 8)], each ideal of A is a product of primary ideals of A.

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