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NEARLY TRIPLY REGULAR HADAMARD DESIGNS AND TOURNAMENTS

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1. Introduction. Let A be a Hadamard 2-$(4\lambda+3, 2\lambda+1, \lambda)$ design. Namely A is a $(0,1)$-matrix of degree $4\lambda+3$, where λ is a positive integer, such that

$$(1) \qquad AA^t = (\lambda+1)I + \lambda J,$$

where t denotes the transposition, and I and J are the identity and all one matrices of degree $4\lambda+3$ respectively. A Hadamard design A is called a Hadamard tournament if A satisfies the following equation

$$(2) \qquad A + A^t + I = J.$$

We choose the mode that each row vector of A is a block, or more precisely the incidence vector of a block. If A is a Hadamard tournament, then a block is an out-neighborhood of a vertex.

A Hadamard design is called nearly triply regular, if $|\alpha \cap \beta \cap \gamma|$ takes only two distinct values μ and ν for any three distinct blocks, α, β and γ, where $|X|$ denotes the number of elements of a finite set X. We assume that $\mu > \nu$. It is known that $|\alpha \cap \beta \cap \gamma|$ takes at least two distinct values. In fact, if $|\alpha \cap \beta \cap \gamma|$ equals a constant μ, then fixing α and β and varying γ, we get the equation $(4\lambda+1)\mu = (2\lambda-1)\lambda$, which is absurd.

The concept of nearly triple regularity is first introduced by M. Herzog and K. B. Reid [2] for Hadamard tournaments. They have shown that Hadamard tournaments of quadratic residue type of orders 7 and 11 are only nearly triply regular Hadamard tournaments with $\nu = 0$. However, its dual concept which is named quasi 3 has been introduced by P. Cameron several years earlier [1]. Furthermore, he showed, in particular, that only quasi 3-Hadamard designs are (i) Hadamard designs of projective geometry over GF(2) type and (ii) Hadamard designs of quadratic residue type of order 11. Since the dual (converse) of a Hadamard tournament is a Hadamard tournament, the result of M. Herzog and K. B. Reid may be regarded as a special case of the result of P. Cameron.

In the present paper we give a more direct and more elementary proof to the mentioned results of P. Cameron based on the nearly triple regularity, even though we have to separate the case $\lambda = 8$ (For this see [3]).

Now we formulate the theorem.

Theorem. (i) *Hadamard designs of projective geometry over* GF(2) *type are the only nearly triply regular Hadamard designs with* $\mu = \lambda$. *These designs are also called of affine geometry over* GF(2) *type, or of group type.*

(ii) *Nearly triply reuglar Hadamard designs with* $\mu < \lambda$ *may exist only when* $\lambda = 2$ *and* 8.

(iii) *Among the designs of* (i) *only the one with* $\lambda = 1$ *is equivalent to a Hadamard tournament.*

2. Hadamard designs of projective geometry over GF(2) type. Let V be an $(n+2)$-dimensional vector space over GF(2) and let PG be the $(n+1)$-dimensional projective space over GF(2) where a point of PG is a 1-dimensional subspace of V. Let P and B be the sets of points and hyperspaces of PG respectively. Then $D_n = (P, B)$ is a Hadamard design of projective geometry over GF(2) type with $\lambda = 2^n - 1$. It is obvious that D_n is nearly triply regular with $\mu = 1$ and $\nu = (\lambda - 1)/2$, and that D_n is self-dual.

Proposition 1. *Let* $D = (P, B)$ *be a nearly triply regular Hadamard* 2-$(4\lambda+3, 2\lambda+1, \lambda)$ *design with* $\mu = \lambda$, *where* P *and* B *denote the sets of points and blocks of* D *respectively. Then* D *is equivalent to* D_n *where* $\lambda = 2^n - 1$.

Proof. The nearly triple regularity with $\mu = \lambda$ implies that for any two distinct blocks α and β there exists a unique block γ such that $\Delta = \alpha \cap \beta = \beta \cap \gamma = \gamma \cap \alpha$. So we define an addition in $B \cup \{P\}$ as follows: $P + P = P$, $\alpha + \alpha = P$, $\alpha + P = \alpha$ and $\alpha + \beta = \gamma$. We notice that this addition is a natural one, namely $\alpha + \beta = (\alpha \cap \beta) \cup (\alpha^c \cap \beta^c)$, where c denotes the complementation. So the associative law holds and $(B \cup \{P\}, +)$ forms an elementary Abelian group of order $4\lambda + 4$. We put $4\lambda + 4 = 2^{n+2}$. Then $\lambda = 2^n - 1$. This shows that D is equivalent to the dual of D_n and, since D_n is self-dual, D is equivalent to D_n.

Remark. Proposition 1 is generalized to a general symmetric design by Blessilda Raposa in Ateneo de Manila University, the Philippines, unaware of the results of P. Cameron [4].

Proposition 2. *Let* D_n *be equivalent to a Hadamard tournament. Then* $n = 1$.

Proof. Assume that $n > 1$. Then for any three distinct blocks α, β and γ, $\alpha \cap \beta \cap \gamma$ is not empty. Let α, β and γ be the out-neighborhoods of vertices a, b and c. Let d be a vertex of $\alpha \cap \beta \cap \gamma$, and consider the out-neighborhood δ of d. Then, since δ is a hyperspace, δ contains either a or b or c. This is a contradiction.

From now on we assume that $\mu < \lambda$.

3. The case where $\mu < \lambda$. Let D be a nearly triply regular Hadamard 2-$(4\lambda+3,\ 2\lambda+1,\ \lambda)$ design with $\mu < \lambda$. Apparently we may assume that

$$\lambda \geqq 3. \tag{3}$$

Let α and β be two distinct blocks of D and put $\Delta = \alpha \cap \beta$. Let c and d be the numbers of blocks γ and δ such that $|D \cap \gamma| = \mu$ and $|D \cap \delta| = \nu$ respectively. Then we have the following equations:

$$c+d = 4\lambda+1, \tag{4}$$

$$c\mu+d\nu = \lambda(2\lambda-1), \tag{5}$$

and

$$c(\mu-1)\mu+d(\nu-1)\nu = (\lambda-2)(\lambda-1)\lambda. \tag{6}$$

Eliminating c and d from (4), (5) and (6) we obtain that

$$(4\lambda+1)\mu\nu = \lambda\{(\mu+\nu-1)(2\lambda-1)-(\lambda-1)(\lambda-2)\}. \tag{7}$$

If $\nu = 0$, then we obtain that $(\mu-1)(2\lambda-1) = (\lambda-1)(\lambda-2)$, which implies that $\lambda = 2$ against (3). So we may put

$$\mu\nu = A\lambda, \tag{8}$$

where A is a positive integer. Then from (7) and (8) we obtain that

$$A(4\lambda+1) = (\mu+\nu-1)(2\lambda-1)-(\lambda-1)(\lambda-2), \tag{9}$$

which implies that

$$12A+3 = (2\lambda-1)(4\mu+4\nu+1-8A-2\lambda). \tag{10}$$

Since $\lambda > \mu > \nu > A$, $12A+3 \leqq 12(\lambda-3)+3 = 12\lambda-33$. Hence from (10) we obtain that

$$4\mu+4\nu+1-8A-2\lambda = 1,\ 3,\ \text{or}\ 5. \tag{11}$$

If $4\mu+4\nu+1-8A-2\lambda = 5$, from (8), (10) and (11) we obtain that

(12) $$6A = 5\lambda - 4,$$
(13) $$6\mu\nu = (5\lambda - 4)\lambda$$

and

(14) $$6\mu + 6\nu = 13\lambda - 2.$$

So 6μ and 6ν are roots of the polynomial $F(X) = X^2 - (13\lambda - 2)X + 6\lambda(5\lambda - 4)$. Since $F(6\lambda) = -12\lambda^2 - 12\lambda < 0$, the larger root of $F(X)$ is greater than 6λ. Since $6\nu < 6\mu < 6\lambda$, this is a contradiction.

If $4\mu + 4\nu + 1 - 8A - 2\lambda = 3$, from (8), (10) and (11) we obtain that

(15) $$2A = \lambda - 1,$$
(16) $$2\mu\nu = (\lambda - 1)$$

and

(17) $$\mu + \nu = \lambda.$$

So 2μ and 2ν are roots of the polynomial $F(X) = X^2 - (3\lambda - 1)X + 2(\lambda - 1)\lambda = (X - 2\lambda)(X - \lambda + 1)$, which implies that $\mu = \lambda$ against our assumption.

Hence we have that $4\mu + 4\nu + 1 - 8A - 2\lambda = 1$. From (8), (10) and (11) we obtain that

(18) $$6A = \lambda - 2,$$
(19) $$6\mu\nu = (\lambda - 2)\lambda$$

and

(20) $$6\mu + 6\nu = 5\lambda - 4.$$

So 6μ and 6ν are roots of the polynomial $F(X) = X^2 - (5\lambda - 4)X + 6\lambda(\lambda - 2) = (X - 3\lambda)(X - 2\lambda + 4)$. Hence we obtain that

(21) $$\mu = \lambda/2 \text{ and } \nu = (\lambda - 2)/3.$$

Then going back to (4) and (5) we obtain that

(22) $$d = 9 - (36/(\lambda + 4)).$$

Hence by (3) we obtain that $\lambda = 8$, 14 or 32. Now we count the number t of trios $\{\alpha, \beta, \gamma\}$ of blocks such that $|\alpha \cap \beta \cap \gamma| = \nu$. For each choice of pairs $\{\alpha, \beta\}$ of blocks there exist d blocks γ such that $|\alpha \cap \beta \cap \gamma| = \nu$. Hence we obtain that

(23) $$t = (4\lambda + 3)(2\lambda + 1)d/3.$$

However, for $\lambda = 14$ and 32 we have that $d = 7$ and 8 respectively. So the right hand side of (23) is not an integer for $\lambda = 14$ and 32.

For $\lambda = 8$ we have that $d = 6$ and $t = 1190$.

This completes the proof of Theorem.

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