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C-TOTALLY REAL SUBMANIFOLDS OF $S^7(1)$ WITH NON-NEGATIVE SECTIONAL CURVATURE

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1. Introduction. In this paper, we study 3-dimensional, compact, minimal C-totally real (connected) submanifolds M in $S^7(1)$. The Sasakian structure on S^7 is recalled in section 2. Fundamental formulas for such submanifolds are given in section 3 ; for more details, see [B], $[\text{Y-K}]_1$, $[\text{Y-K}]_2$.

Let K denote the sectional curvature function on M. Then Van Lindt, Verheyen and Verstraelen proved the following : if $K > 0$, then M is totally geodesic in S^7. The main purpose of this paper is to prove the two following results.

Corollary 4.1. *If M is a 3-dimensional, compact, minimal, C-totally real submanifold of S^7 and if all the sectional curvatures K of M satisfy $0 \leqq K \leqq 1$ then either $K \equiv 1$ (M is totally geodesic) or $K \equiv 0$ (M is flat).*

Main theorem. *Let $x: M^3 \to S^7$ be a C-totally real, minimal immersion of a 3-dimensional compact Riemannian manifold M. If the sectional curvatures K of M satisfy $K \geqq 0$, then either*

(i) *M is simply connected and x is congruent to $i: S^3 \to S^7$ (i.e. M is totally geodesic in S^7),*

(ii) *M is a covering of T^3 with covering map π and x is congruent to $j \circ \pi: M \to S^7$,*

(iii) *M is a covering of $S^1(\sqrt{3}) \times S^2(\sqrt{3}/2)$ with covering map π and x is congruent to $k \circ \pi: M \to S^7$,*

where the immersions i, j and k are defined in section 5.

2. The Sasakian structure on S^{2n+1}. We consider the unit sphere S^{2n+1} in $\mathbb{C}^{n+1}$. Using the natural complex structure J of $\mathbb{C}^{n+1}$, we define a tangent vector field ξ, a 1-form η, and a (1, 1) tensor field ϕ on S^{2n+1} as follows :

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$<\,,\,>$ is the induced metric from $\mathbb{C}^{n+1}$ on S^{2n+1} (So S^{2n+1} has constant sectional curvature 1)

$\xi_p = J_p N$, where N is the unit normal in p, for all $p \in S^{2n+1}$

$\eta(X) = <X, \xi>$

$\phi = s \circ J$,

where s denotes the orthogonal projection from $T_p\mathbb{C}^{n+1}$ on T_pS^{2n+1}. Using these definitions, we obtain for all tangent vector fields X and Y on S^{2n+1} that

$$\phi^2 X = -X + \eta(X)\xi, \tag{2.1}$$

$$\phi\xi = 0, \tag{2.2}$$

$$\eta(\phi X) = 0, \tag{2.3}$$

$$\eta(\xi) = 1, \tag{2.4}$$

$$N(X, Y) = -2\, d\eta \otimes \xi, \tag{2.5}$$

$$<\phi X, \phi Y> = <X, Y> - \eta(X)\eta(Y), \tag{2.6}$$

where N is defined by $N(X, Y) = [\phi X, \phi Y] + \phi^2[X, Y] - \phi[X, \phi Y] - \phi[\phi X, Y]$. These formulas imply that $(\phi, \xi, \eta, <\,,\,>)$ determines a Sasakian structure on S^{2n+1} with constant ϕ-sectional curvature 1. Therefore, we also have ([B], [Y-K]$_2$)

$$\tilde{\nabla}_X \xi = \phi X \tag{2.7}$$

and

$$(\tilde{\nabla}_X \phi)(Y) = -<X, Y>\xi + \eta(Y)X, \tag{2.8}$$

where $\tilde{\nabla}$ denotes the Levi Civita connection of $<\,,\,>$ and X, Y are tangent vector fields on S^{2n+1}.

3. C-totally real submanifolds of S^{2n+1}. A Riemannian manifold M^m, isometrically immersed in S^{2n+1}, is called a C-totally real submanifold of S^{2n+1}, if ξ is a normal vector field on M. A direct consequence of this definition is that $\phi(TM) \subset T^\perp M$ (i.e. that M is an anti-invariant submanifold of S^{2n+1}), since for $X, Y \in \mathfrak{X}(M)$

$$\begin{aligned} <\phi X, Y> &= (1/2)\{<\phi X, Y> - <X, \phi Y>\} \\ &= (1/2)\{<\tilde{\nabla}_X \xi, Y> - <X, \tilde{\nabla}_Y \xi>\} \\ &= -(1/2)<\xi, [X, Y]> = 0. \end{aligned}$$

In this paper we consider the case dim $M = n$. We denote the Levi Civita connection of M by ∇. The formulas of Gauss and Weingarten are then given by

$$(3.1) \qquad \tilde{\nabla}_X Y = \nabla_X Y + h(X, Y)$$

and

$$(3.2) \qquad \tilde{\nabla}_X \alpha = -A_\alpha X + \nabla_X^\perp \alpha,$$

where X and Y are tangent vector fields on M and α is a normal vector field on M. The second fundamental form h is related to A_α by

$$(3.3) \qquad <h(X, Y), \alpha> = <A_\alpha X, Y>.$$

From (3.1) and (3.2), we find that

$$(3.4) \qquad \nabla_X^\perp \phi Y = \phi \nabla_X Y - <X, Y>\xi$$

and

$$(3.5) \qquad \phi h(X, Y) = -A_{\phi Y} X.$$

If we denote the curvature tensors of ∇ and $\nabla^\perp$ by R and $R^\perp$, respectively, then the equations of Gauss, Codazzi and Ricci are given by

$$(3.6) \qquad R(X, Y)Z = <Y, Z>X - <X, Z>Y + A_{h(Y,Z)}X - A_{h(X,Z)}Y$$

$$(3.7) \qquad (\nabla h)(X, Y, Z) = (\nabla h)(Y, X, Z)$$

$$(3.8) \qquad <R^\perp(X, Y)\alpha, \beta> = <[A_\alpha, A_\beta]X, Y>$$

where X, Y, Z (respectively α, β) are tangent (respectively normal) vector fields on M and (∇h) is defined by $(\nabla h)(X, Y, Z) = \nabla_X^\perp h(Y, Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z)$. From (2.7), (3.2) and (3.3), it immediately follows that

$$(3.9) \qquad <h(X, Y), \xi> = 0.$$

By means of a straightforward calculation, we also obtain for tangent vector fields X, Y, Z on M that

$$(3.10) \qquad <h(X, Y), \phi Z> = <h(X, Z), \phi Y>,$$

$$(3.11) \qquad <(\nabla h)(X, Y, Z), \xi> = -<h(Y, Z), \phi X>.$$

Finally, we will use the following proposition from [V-V].

Proposition 3.1. *If M is an n-dimensional compact, minimal, C-totally* *real submanifold of* S^{2n+1} *and if all the sectional curvatures* K *of* M *satisfy* $K \geqq 0$, *then*:

$$(1) \quad (\nabla h)(u, v, w) = \langle (\nabla h)(u, v, w), \xi(p)\rangle \xi(p),$$

and

$$(2) \quad R(v, A_{\phi v}v, A_{\phi v}v, v) = 0$$

for all $p \in M$ *and* $u, v, w \in T_pM$.

4. 3-dimensional C**-totally real submanifolds of** S^7**.** In this section, M will always denote a 3-dimensional, compact, minimal, C-totally real submanifold of S^7. We now choose an orthonormal basis of T_pM, at a point p of M, in the following way. We define a function f on UM_p by

$$f(u) = \langle h(u, u), \phi(u)\rangle,$$

for $u \in UM_p$. Let e_1 be an element of UM_p in which the function f attains an absolute maximum. By using (3.10), we see that this implies that $\langle h(e_1, e_1), \phi(u)\rangle = 0$, for $u \in UM_p$ with $\langle u, e_1\rangle = 0$. So e_1 is an eigenvector of $A_{\phi e_1}$. Now we consider the function f_1 which is nothing else but the function f restricted to B, where $B = \{u \in UM_p | \langle u, e_1\rangle = 0\}$. If f_1 is constant, then we choose e_2 and e_3 as eigenvectors of $A_{\phi e_1}$. If f_1 is not constant, we choose e_2 as an element of B in which the function f_1 attains its absolute maximum. Therefore $\langle h(e_2, e_2), \phi(u)\rangle = 0$, where u is orthogonal to both e_1 and e_2. Finally we take e_3 orthonormal to e_1 and e_2. Since, we can still change the sign of e_3, we also may assume that $\langle h(e_1, e_2), \phi e_3\rangle \geqq 0$. Using this, (3.9), (3.10) and the minimality of M, we obtain the following expressions for the second fundamental form h at the point p.

$$\begin{aligned}
h(e_1, e_1) &= a\phi e_1, \\
h(e_2, e_2) &= b\phi e_1 + d\phi e_2, \\
h(e_3, e_3) &= -(a+b)\phi e_1 - d\phi e_2, \\
h(e_1, e_2) &= b\phi e_2 + c\phi e_3, \\
h(e_1, e_3) &= -(a+b)\phi e_3 + c\phi e_2, \\
h(e_2, e_3) &= c\phi e_1 - d\phi e_3,
\end{aligned}$$

where $a \geqq d \geqq 0$, $c \geqq 0$ and $b \in \mathbb{R}$. Since $\{e_1, e_2, e_3\}$ is an orthonormal basis of T_pM, any vector $v \in T_pM$ can be written as $v = v_1 e_1 + v_2 e_2 + v_3 e_3$, where $v_1, v_2, v_3 \in \mathbb{R}$. Therefore, using the Gauss equation, we find after a very long but straightforward computation* that the second condition of Proposition 3.1 is equivalent to the following equations in a, b, c and d.

(4.1) $-2a^4-7a^3b-9a^2b^2-a^2c^2-2a^2d^2+a^2-5ab^3-2abc^2-7abd^2$
$+2ab-b^4-b^2c^2-5b^2d^2+b^2-c^2d^2-2d^4+d^2=0,$

(4.2) $c(3a^3+7a^2b+5ab^2+ac^2-a+b^3+bc^2+bd^2-b)=0,$

(4.3) $5a^3b+11a^2b^2-25a^2c^2+6a^2d^2+a^2+7ab^3-28abc^2+26abd^2+b^4$
$-3b^2c^2+22b^2d^2-b^2-4c^4+14c^2d^2+4c^2+12d^4-6d^2=0,$

(4.4) $c(-5a^2b-5ab^2+5ac^2-a-2bd^2)=0,$

(4.5) $-4a^2b^2-3ab^3+22abc^2-15abd^2-2ab+b^4-3b^2c^2-21b^2d^2-b^2$
$-4c^4-33c^2d^2+4c^2-18d^4+9d^2=0,$

(4.6) $bc(2ab-b^2-c^2-3d^2+1)=0,$

(4.7) $b^2(ab-b^2-c^2+1)=0,$

(4.8) $cd(3ab+3b^2+c^2+3d^2-1)=0,$

(4.9) $d(6a^3+21a^2b+30ab^2+9ac^2+6ad^2-5a+15b^3+3bc^2+12bd^2$
$-7b)=0,$

(4.10) $cd(-6a^2-15ab-15b^2-11c^2-12d^2+7)=0,$

(4.11) $d(-11a^2b-31ab^2-11ac^2-18ad^2+7a-24b^3-32bc^2-36bd^2$
$+16b)=0,$

(4.12) $cd(2ab+2b^2+8c^2+9d^2-4)=0,$

(4.13) $bd(-ab+b^2+5c^2-1)=0,$

(4.14) $12a^4+38a^3b+44a^2b^2+14a^2c^2+6a^2d^2-6a^2+22ab^3+14abc^2$
$+17abd^2-10ab+4b^4+7b^2d^2-4b^2-4c^4-25c^2d^2+4c^2+d^2=0,$

(4.15) $c(-12a^3-23a^2b-19ab^2-11ac^2-6ad^2+7a-8b^3-8bc^2-20bd^2$
$+8b)=0,$

(4.16) $-9a^3b-21a^2b^2+27a^2c^2-21a^2d^2+a^2-24ab^3-63abd^2+12ab$
$-12b^4-45b^2d^2+12b^2+12c^4+27c^2d^2-12c^2+d^2=0,$

(4.17) $c(9a^2b+5ab^2-3ac^2+6ad^2-a+8b^3+8bc^2+28bd^2-8b)=0,$

*For this computation, the authors used a computer.

(4.18) $2a^2b^2-6ab^3-14abc^2+abd^2+2ab+4b^4-b^2d^2-4b^2-4c^2 -17c^2d^2+4c^2+d^2=0,$

(4.19) $cd(-7ab-3b^2+5c^2-1)=0,$

(4.20) $d(-18a^3-53a^2b-61ab^2-11ac^2+7a-18b^3+30bc^2+2b)=0,$

(4.21) $cd(6a^2+17ab+21b^2-3c^2-1)=0,$

(4.22) $d(-a^2b+3ab^2+5ac^2-a-2b^3-18bc^2+2b)=0,$

(4.23) $-18a^4-51a^3b-53a^2b^2-33a^2c^2+9a^2-24ab^3-24abc^2+12ab -4b^4-8b^2c^2+4b^2-4c^4+4c^2=0,$

(4.24) $ac(9a^2+8ab+8b^2+8c^2-4)=0,$

(4.25) $a^3b-5a^2b^2-17a^2c^2+a^2+8ab^3+8abc^2-4ab-4b^4-8b^2c^2+4b^2 -4c^4+4c^2=0.$

In order to solve these equations, we consider the following cases :

case 1: $a\neq 0$, $b\neq 0$, $c\neq 0$, $d\neq 0$. A contradiction follows if we compare (4.7) with (4.13).

case 2: $a\neq 0$, $b\neq 0$, $c\neq 0$, $d=0$. In this case, f_1 is identically zero. Therefore, we took e_2 as an eigenvector of $A_{\phi e_1}$. This implies that $c=0$.

case 3: $a\neq 0$, $b\neq 0$, $c=0$. Then the equation (4.7) becomes

(4.26) $$a=(b^2-1)/b.$$

Substituting this value for a in (4.23) gives

$$b^2=1/3 \text{ or } b^2=3/5.$$

Using the fact that $a\geqq 0$, we deduce from (4.26) that either

$$b=-1/\sqrt{3} \text{ and } a=2/\sqrt{3},$$

or

$$b=-\sqrt{3}/\sqrt{5} \text{ and } a=2/(\sqrt{3}\sqrt{5}).$$

So, we have to consider two subcases.

subcase 3a: $a=2/\sqrt{3}$, $b=-1/\sqrt{3}$, $c=0$. From (4.1) we deduce that either $d=0$ or $d=\sqrt{2}/\sqrt{3}$. Then, after a straightforward calculation one sees that in both cases all the other equations are also satisfied.

subcase 3b: $a=2/(\sqrt{3}\sqrt{5})$, $b=-3/\sqrt{5}$, $c=0$. Here, it follows from (4.16) that $d = 2/\sqrt{15}$. If we put $u = (\sqrt{2}/2)(e_2 - e_1)$, we see that $<h(u, u), \phi(u)> = (\sqrt{2}/4)(3\sqrt{3}/\sqrt{5}) > 2/\sqrt{15}$. This implies that e_1 is not an absolute maximum.

case 4: $a \neq 0$, $b = 0$, $c = 0$. This is not possible, since (4.25) implies $a = 0$.

case 5: $a \neq 0$, $b = 0$, $c \neq 0$, $d = 0$. Applying the same argument as in case 2, we obtain a contradiction.

case 6: $a \neq 0$, $b = 0$, $c \neq 0$, $d \neq 0$. First, we deduce from (4.22) that $c^2 = 1/5$. Then, it follows from (4.21) that $a = 2/\sqrt{15}$. From (4.8) we then find that $d = 2/\sqrt{15}$. Now, we put $u = -1/\sqrt{5}\{e_1 + e_2 - \sqrt{3}e_3\}$. Then we notice that $<h(u, u), \phi(u)> = (-1/5\sqrt{5})(-50/\sqrt{15}) > 2/\sqrt{15}$.

case 7: $a = 0$. This implies that f is identically zero. By linearization, we then deduce that also $b = c = d = 0$.

By combining this with Proposition 3.1, we immediately obtain the following lemma.

Lemma 4.1. *If M is a 3-dimensional, compact, minimal, C-totally real submanifold of S^7 and if all the sectional curvatures K of M satisfy $K \geqq 0$, then, for each point p of M, there exists an orthonormal basis $\{e_1, e_2, e_3\}$ of T_pM such that either*

(4.27) (i) $h(e_1, e_1) = h(e_2, e_2) = h(e_3, e_3) = 0,$
$h(e_1, e_2) = h(e_1, e_3) = h(e_2, e_3) = 0.$

(4.28) (ii) $h(e_1, e_1) = (2/\sqrt{3})\phi e_1,$
$h(e_2, e_2) = -(1/\sqrt{3})\phi e_1 + (\sqrt{2}/\sqrt{3})\phi e_2,$
$h(e_3, e_3) = -(1/\sqrt{3})\phi e_1 - (\sqrt{2}/\sqrt{3})\phi e_2,$
$h(e_1, e_2) = -(1/\sqrt{3})\phi e_2,$
$h(e_1, e_3) = -(1/\sqrt{3})\phi e_3,$
$h(e_2, e_3) = -(\sqrt{2}/\sqrt{3})\phi e_3,$

or

(4.29) (iii) $h(e_1, e_1) = (2/\sqrt{3})\phi e_1,$
$h(e_2, e_2) = h(e_3, e_3) = -(1/\sqrt{3})\phi e_1,$
$h(e_1, e_3) = -(1/\sqrt{3})\phi e_3,$
$h(e_1, e_2) = -(1/\sqrt{3})\phi e_2,$

$$h(e_2, e_3) = 0.$$

Let M be like in Lemma 4.1. Then we have the following proposition.

Proposition 4.1. *Let $p \in M$. Then:*

(a) *if* (4.27) *holds*, $K(p) \equiv 1$,

(b) *if* (4.28) *holds*, $K(p) \equiv 0$,

(c) *if* (4.29) *holds, then* $0 \leqq K(p) \leqq 4/3$, *where* $K(p) = 0$ *for every plane through* e_1 *and* $K(p) = 4/3$ *only for the plane through* e_2 *and* e_3.

Proof.

(a) In this case, $h \equiv 0$, so p is a totally geodesic point. From the Gauss equation we obtain that $K(p) \equiv 1$.

(b) From the Gauss equation and (4.28) we find that

$$R(e_1, e_2)e_2 = R(e_1, e_3)e_3 = R(e_2, e_3)e_3 = 0$$
$$R(e_1, e_2)e_3 = R(e_2, e_3)e_1 = R(e_3, e_1)e_2 = 0.$$

Linearization then implies that $R_p \equiv 0$ and therefore $K(p) \equiv 0$.

(c) From the Gauss equation and (4.29), we obtain that

$$R(e_1, e_2)e_3 = R(e_2, e_3)e_1 = R(e_3, e_1)e_2 = 0,$$
$$R(e_1, e_2)e_2 = R(e_1, e_3)e_3 = 0,$$
$$R(e_2, e_3)e_3 = (4/3)e_2.$$

Let σ be any plane section of T_pM. Then we can find an orthonormal basis $\{X, Y\}$ of σ such that $X = \cos\theta\, e_2 + \sin\theta\, e_3$ and $Y = \sin\varphi\, e_1 - \cos\varphi \sin\theta\, e_2 + \cos\varphi\cos\theta\, e_3$ where $\theta, \varphi \in \mathbb{R}$. Then

$$\begin{aligned} R(X, Y, Y, X) &= \cos^2\theta R(e_2, Y, Y, e_2) + 2\cos\theta\sin\theta R(e_2, Y, Y, e_3) \\ &\quad + \sin^2\theta R(e_3, Y, Y, e_3) \\ &= (4/3)\cos^4\theta\cos^2\varphi + (8/3)\cos^2\varphi\cos^2\theta\sin^2\theta \\ &\quad + (4/3)\sin^4\theta\cos^2\varphi = (4/3)\cos^2\varphi. \end{aligned}$$

Let M be as in Lemma 4.1. The connectedness of M and Proposition 4.1 imply that if M satisfies condition (4.27) (respectively (4.28), (4.29)) in a point p of M, then M satisfies condition (4.27) (respectively (4.28), (4.29)) in every point of M. The next statement follows then easily from Proposition 4.1.

Corollary 4.1. *If M is a 3-dimensional, compact, minimal, C-totally* *real submanifold of S^7 and if all the sectional curvatures K of M satisfy either $0 \leqq K \leqq 1$ or $0 \leq K < 4/3$, then either $K \equiv 1$ (M is totally geodesic) or $K \equiv 0$ (M is flat) on M.*

In the following proposition, we study more closely the case (c) of Proposition 4.1.

Proposition 4.2. *Let M be a 3-dimensional, compact, minimal, C-totally real submanifold of S^7 with K not constant and satisfying $K \geqq 0$. Then there exists globally a unique tangent vector field E_1 and locally tangent vector fields E_2 and E_3 such that*

(a) $\{E_1, E_2, E_3\}$ *is a local orthonormal frame,*
(b) *For any $p \in M$, f attains its maximum value at $E_1(p)$,*
(c) $h(E_1, E_1) = (2/\sqrt{3})\phi E_1$,
$h(E_2, E_2) = h(E_3, E_3) = -(1/\sqrt{3})\phi E_1$,
$h(E_1, E_3) = -(1/\sqrt{3})\phi E_3$,
$h(E_1, E_2) = -(1/\sqrt{3})\phi E_2$,
$h(E_2, E_3) = 0$.
(d) $\nabla_{E_1} E_1 = \nabla_{E_2} E_1 = \nabla_{E_3} E_1 = 0$.

Proof. Since $K \geqq 0$ and K is not constant, it immediately follows from Proposition 4.1 and Lemma 4.1 that the vector field $E_1(p)$, where $E_1(p)$ is the unique point at which f attains its maximum at p, is well defined on the whole of M and is also differentiable. This proves (a), (b) and (c). In order to prove (d), we use the first part of Proposition 3.1. From

$$(\nabla h)(E_1, E_1, E_1) \equiv 0 \pmod{\xi},$$

we obtain

$$\nabla_{E_1} E_1 = 0.$$

The other claims follow then from

$$(\nabla h)(E_2, E_1, E_1) \equiv 0 \pmod{\xi} \text{ and } (\nabla h)(E_3, E_1, E_1) \equiv 0 \pmod{\xi}.$$

Theorem 4.1. *Let M be a 3-dimensional, compact, minimal, C-totally real submanifold of S^7 with sectional curvatures K satisfying $K \geqq 0$. Let $\tilde{M}$ be the universal covering of M. Then either*

$$\tilde{M} \cong S^3(1),$$

or

$$\tilde{M} \cong \mathbb{R}^3$$

or

$$\tilde{M} \cong \mathbb{R}\times S^2(\sqrt{3}/2),$$

where $S^m(r)$ denotes the m-dimensional sphere with radius r.

Proof. If the sectional curvature K of M is constant, we immediately obtain that $\tilde{M} \cong \mathbb{R}^3$ or $\tilde{M} \cong S^3(1)$. Therefore, we may assume that K is not constant. Then, we know from Proposition 4.2 that there exists a globally defined vector field E_1 on M. Now we define 2 distributions T_0 and T_1 on M by

$$\begin{aligned} T_0 &: p \mapsto T_0|_p = \mathrm{vect}\{E_1(p)\}, \\ T_1 &: p \mapsto T_1|_p = \{v \in T_pM| <v, e_1> = 0\}. \end{aligned}$$

From (d) of Proposition 4.2 we know that the distributions T_0 and T_1 satisfy $\nabla_{T_0} T_0 \subset T_0$, $\nabla_{T_1} T_0 \subset T_0$. From this we easily deduce that also $\nabla_{T_0} T_1 \subset T_1$ and $\nabla_{T_1} T_1 \subset T_1$. Let $\tilde{T}_0$, $\tilde{T}_1$ denote the corresponding distributions on $\tilde{M}$. Clearly $\tilde{T}_0$ and $\tilde{T}_1$ also satisfy $\nabla_{\tilde{T}_0} \tilde{T}_0 \subset \tilde{T}_0$, $\nabla_{\tilde{T}_1} \tilde{T}_1 \subset \tilde{T}_1$, $\nabla_{\tilde{T}_0} \tilde{T}_1 \subset \tilde{T}_1$ and $\nabla_{\tilde{T}_1} \tilde{T}_0 \subset \tilde{T}_0$. Then, we know from the de Rham decomposition theorem that

$$\tilde{M} \cong \mathbb{R}\times S^2(\sqrt{3}/2).$$

This completes the proof.

From the proof of Theorem 4.1, we see that we can choose the orientation of $\mathbb{R}\times S^2(\sqrt{3}/2)$ such that the vectorfield $\tilde{E}_1$ on $\mathbb{R}\times S^2(\sqrt{3}/2)$ is given by

$$\tilde{E}_1(\alpha, \beta) = (1, 0)_{(\alpha,\beta)}. \tag{4.30}$$

In the next section, we will also need the following proposition about flat, C-totally real submanifolds.

Proposition 4.3. *Let M be a 3-dimensional, flat, compact, minimal C-totally real submanifold of S^7. Then there exists globally an orthonormal frame $\{E_1, E_2, E_3\}$ such that*

(a) *For any $p \in M$, f attains its maximum value at $E_1(p)$ and f_1*

attains its maximum value at $E_2(p)$;

(b) $h(E_1, E_1) = (2/\sqrt{3})\phi E_1$,
$h(E_2, E_2) = -(1/\sqrt{3})\phi E_1 + (\sqrt{2}/\sqrt{3})\phi E_2$,
$h(E_3, E_3) = -(1/\sqrt{3})\phi E_1 - (\sqrt{2}/\sqrt{3})\phi E_2$,
$h(E_1, E_2) = -(1/\sqrt{3})\phi E_2$,
$h(E_1, E_3) = -(1/\sqrt{3})\phi E_3$,
$h(E_2, E_3) = -(\sqrt{2}/\sqrt{3})\phi E_3$

(c) $\nabla_{E_i} E_j = 0$ *for* $i, j \in \{1, 2, 3\}$.

Furthermore, we can choose Euclidean coordinates on $\mathbb{R}^3$, *the universal covering of* M, *such that the corresponding orthonormal frame* $\{\tilde{E}_1, \tilde{E}_2, \tilde{E}_3\}$ *on* $\mathbb{R}^3$ *has the following form*:

$$\tilde{E}_1(x, y, z) = (1, 0, 0)_{(x,y,z)},$$
$$\tilde{E}_2(x, y, z) = (0, 1, 0)_{(x,y,z)},$$
$$\tilde{E}_3(x, y, z) = (0, 0, 1)_{(x,y,z)}.$$

Proof. Let $p \in M$. First, we notice that for every basis, chosen as in the beginning of this section (4.28) holds.

Next, we can write $v \in UM_p$ as $v = \alpha e_1 + \beta e_2 + \gamma e_3$, where $\alpha^2 + \beta^2 + \gamma^2 = 1$. A straightforward computation then shows that

$$\begin{aligned} h(v, v) = &((2/\sqrt{3})\alpha^2 - (1/\sqrt{3})\beta^2 - (1/\sqrt{3})\gamma^2)\phi e_1 \\ &+((\sqrt{2}/\sqrt{3})\beta^2 - (\sqrt{2}/\sqrt{3})\gamma^2 - (2/\sqrt{3})\alpha\beta)\phi e_2 \\ &+(-(2/\sqrt{3})\alpha\gamma - (2\sqrt{2}/\sqrt{3})\beta\gamma)\phi e_3 \end{aligned}$$

and

$$\begin{aligned} \| h(v, v)\|^2 &= (4/3)\alpha^4 + \beta^4 + \gamma^4 + 2\beta^2\gamma^2 + 4\sqrt{2}\,\alpha\beta\gamma^2 - (4\sqrt{2}/3)\alpha\beta^3 \\ &= (4/3)(\alpha^2+\beta^2+\gamma^2)^2 - (1/3)(\beta^4+\gamma^4+2\beta^2\gamma^2) \\ &\quad -(1/3)(4\sqrt{2}\,\alpha(\beta^3 - 3\beta\gamma^2)) - (1/3)(8\alpha^2(\beta^2+\gamma^2)). \end{aligned}$$

Then we define

$$(4.31)\quad \begin{aligned} h(\alpha, \beta, \gamma) &= 8\alpha^2(\beta^2+\gamma^2) + 4\sqrt{2}\,\alpha(\beta^3 - 3\beta\gamma^2) + (\beta^4+\gamma^4+2\beta^2\gamma^2), \\ g(\beta, \gamma) &= -8\gamma^2(3\beta^2 - \gamma^2)^2. \end{aligned}$$

Now, we look at h as a quadratic function in α with discriminant g. This implies that $h(\alpha, \beta, \gamma) \geqq 0$ and $h(\alpha, \beta, \gamma) = 0$ only if either

$$(4.32)\qquad \gamma = 0 \text{ and } h(\alpha, \beta, 0) = 0$$

or

(4.33) $$\gamma^2 = 3\beta^2 \text{ and } h(\alpha, \beta, \pm\sqrt{3}\beta) = 0.$$

From (4.31) it is immediately clear that (4.32) is equivalent with

(4.34) $$\begin{aligned} &\gamma = 0\ ;\ \beta = 0\ ;\ \alpha = \pm 1 \\ &\gamma = 0\ ;\ \beta = -2\sqrt{2}\alpha\ ;\ \alpha = \pm 1/3 \end{aligned}$$

and (4.33) is equivalent with either (4.34) or

$$\alpha = \pm 1/3\ ;\ \beta = \sqrt{2}\alpha\ ;\ \gamma = \pm\sqrt{6}\alpha.$$

This implies that $\| h(v, v)\|^2 \leqq 4/3$ and $\| h(v, v)\|^2 = 4/3$ if and only if either

$$v = \pm e_1,$$

or

$$v = \pm((1/3)e_1 + (\sqrt{2}/3)e_2 \pm (\sqrt{6}/3)e_3),$$

or

$$v = \pm((1/3)e_1 - (2\sqrt{2}/3)e_2).$$

From this, we see that e_1, $-((1/3)e_1 + (\sqrt{2}/3)e_2 \pm (\sqrt{6}/3)e_3)$, $-(1/3)e_1 + (2\sqrt{2}/3)e_2$ are the unique vectors in which f attains its maximal value. Therefore, we can define a differentiable vectorfield E_1 on the whole of M such that f attains an absolute maximum in $E_1(p)$. As a matter of fact, we have 4 possible choices for E_1.

Now, we put $u = \cos\theta e_2 + \sin\theta e_3$. Then

$$\begin{aligned} \langle h(u, u), \phi u\rangle &= (\sqrt{2}/\sqrt{3})\cos^3\theta - (3\sqrt{2}/\sqrt{3})\sin^2\theta\cos\theta \\ &= (\sqrt{2}/\sqrt{3})\cos\theta\{\cos^2\theta - 3\sin^2\theta\} \\ &= (\sqrt{2}/\sqrt{3})\cos\theta\{4\cos^2\theta - 3\}. \end{aligned}$$

From this, we see that $\langle h(u, u), \phi(u)\rangle$ is maximal if and only if either

$$u = e_2 \text{ or } u = -(1/2)e_2 \pm (\sqrt{3}/2)e_3.$$

This proves (a) and (b). In order to prove (c), we use the first part of Proposition 3.1. From

$$(\nabla h)(E_i, E_1, E_1) \equiv (\nabla h)(E_i, E_2, E_2) \equiv 0 \pmod{\xi},$$

we see that $\nabla_{E_i}E_1 = \nabla_{E_i}E_2 = \nabla_{E_i}E_3 = 0$.

The rest of the proof follows then immediately from the de Rham decomposition theorem.

5. Some examples. In this section, we give three examples of C-totally real, minimal, 3-dimensional submanifolds of S^7 satisfying $K \geq 0$. Using the results of section 4, we then prove that these examples are basically the only ones.

Example 5.1. If i denotes the inclusion map of

$$S^3 = \{x \in S^7(1) \mid x = x_1 e_1 + x_3 e_3 + x_5 e_5 + x_7 e_7\},$$

where $\{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8\}$ is the standard basis of $\mathbb{R}^8$, then $i: S^3 \to S^7$ is a C-totally real, totally geodesic embedding.

Example 5.2. [Y-K]$_1$ Let $\tilde{j}: \mathbb{R}^3 \to S^7$ be a minimal immersion represented by

$$\begin{aligned} &\tilde{j}(u_1, u_2, u_3) \\ &\quad = (1/2)(\cos u_1, \sin u_1, \cos u_2, \sin u_2, \cos u_3, \sin u_3, \cos u_4, \sin u_4), \end{aligned}$$

where $u_4 = -(u_1 + u_2 + u_3)$. Then $\tilde{j}(\mathbb{R}^3)$ is a compact, flat minimal, C-totally real submanifold of S^7. We will denote $\tilde{j}(\mathbb{R}^3) = T^3$ and the embedding from T^3 into S^7 by j.

Example 5.3. Let $\tilde{k}: \mathbb{R} \times S^2(\sqrt{3}/2) \to S^7: (u, y_1, y_2, y_3) \mapsto (x_1, x_2, \ldots, x_8)$, where $y_1^2 + y_2^2 + y_3^2 = 3/4$ and

$$\begin{cases} x_1 = (1/4)(\cos \sqrt{3}\,u + 2\sqrt{3}\,y_3 \cos(1/\sqrt{3})u), \\ x_2 = (1/4)(\sin \sqrt{3}\,u - 2\sqrt{3}\,y_3 \sin(1/\sqrt{3})u), \\ x_3 = (\sqrt{3}/4)(\sin \sqrt{3}\,u + (2/\sqrt{3})y_3 \sin(1/\sqrt{3})u), \\ x_4 = (\sqrt{3}/4)(-\cos \sqrt{3}\,u + (2/\sqrt{3})y_3 \cos(1/\sqrt{3})u), \\ x_5 = y_2 \cos(1/\sqrt{3})u, \\ x_6 = -y_2 \sin(1/\sqrt{3})u, \\ x_7 = y_1 \cos(1/\sqrt{3})u, \\ x_8 = -y_1 \sin(1/\sqrt{3})u. \end{cases}$$

Furthermore, we consider the standard product metric on $\mathbb{R} \times S^2(\sqrt{3}/2)$. Then, k is a C-totally real, isometric, minimal immersion from $\mathbb{R} \times S^2(\sqrt{3}/2)$ into S^7. Furthermore, we see that

$$\tilde{k}(u, y_1, y_2, y_3) = \tilde{k}(u', y_1', y_2', y_3')$$

if and only if

$$u \equiv u' \pmod{2\pi\sqrt{3}}$$

and

$$y_i = y_i' \qquad \text{for } i = 1, 2, 3.$$

Therefore, we obtain an embedding k from $S^1(\sqrt{3}) \times S^2(\sqrt{3}/2)$ into $S^7(1)$.

Main theorem. *Let $x: M^3 \to S^7$ be a C-totally real, isometric, minimal immersion of a 3-dimensional compact Riemannian manifold M. If the sectional curvatures K of M satisfy $K \geqq 0$, then either*

(i) *M is simply connected and x is congruent to $i: S^3 \to S^7$ (i.e. M is totally geodesic in S^7)*

or

(ii) *M is a covering of T^3 with covering map π and x is congruent to $j \circ \pi: M \to S^7$.*

or

(iii) *M is a covering of $S^1(\sqrt{3}) \times S^2(\sqrt{3}/2)$ with covering map π and x is congruent to $k \circ \pi: M \to S^7$.*

Proof. By Theorem 4.1 we only have to consider the 3 following cases.

case 1: The universal covering of M is S^3. In this case M is totally geodesic in S^7. This completes the proof in this case.

case 2: The universal covering of M is $\mathbb{R}^3$. Then, we have the following diagram:

$$\begin{array}{ccc} & \overset{\pi_1}{\nearrow} & M^3 \xrightarrow{x} S^7 \\ \mathbb{R}^3 & & \\ & \underset{\pi_2}{\searrow} & T^3 \xrightarrow{j} S^7. \end{array}$$

Take a point $p \in \mathbb{R}^3$ and a small neighbourhood of p such that $\pi_1|_U$ and $\pi_2|_U$ are injective. So, we obtain an isometry $\pi: \pi_1(U) \to \pi_2(U)$. By Proposition 4.3, we may assume that $\pi_*(E_i) = E_i^*$, where $i \in \{1, 2, 3\}$ and $\{E_1, E_2, E_3\}$ (respectively $\{E_1^*, E_2^*, E_3^*\}$) is chosen on M (resp. T^3) as is indicated in Proposition 4.3. Then, we define a map $\tilde{\pi}: N\pi_1(U) \to N\pi_2(U)$ by

$$\tilde{\pi}(\phi X) = \phi \pi_* X \text{ and } \tilde{\pi}(\xi) = \xi \circ \pi.$$

Using the fact that both x and j are C-totally real immersions, we see that we can apply the rigidity theorem ([Sp], p. 72—80). Therefore, the following diagram commutes :

$$\begin{array}{ccccc} & \overset{\pi_1}{\nearrow} & M & \xrightarrow{x} & S^7 \\ U & & \odot & & \downarrow A \\ & \underset{\pi_2}{\searrow} & T & \xrightarrow{j} & S^7, \end{array}$$

where A is an isometry of S^7. Since minimal immersions from an analytic manifold in spheres are analytic, this implies

$$\begin{array}{ccccc} & \overset{\pi_1}{\nearrow} & M & \xrightarrow{x} & S^7 \\ \mathbb{R}^3 & & \odot & & \downarrow A \\ & \underset{\pi_2}{\searrow} & T & \xrightarrow{j} & S^7 \end{array}$$

In particular, since j is an embedding, we obtain that the following map

$$j^{-1} \circ A \circ x : \ M \to T$$

is a covering map of T.

case 3: The universal covering of M is $\mathbb{R} \times S^2(\sqrt{3}/2)$. By using Proposition 4.2 and (4.30), we obtain in the same way as in the previous case that the following diagram commutes :

$$\begin{array}{ccccc} & \overset{\pi_1}{\nearrow} & M \xrightarrow{x} S^7 & \searrow^{A} & \\ \mathbb{R} \times S^2(\sqrt{3}/2) & & \odot & & \\ & \underset{\pi_2}{\searrow} & S^1(\sqrt{3}) \times S^2(\sqrt{3}/2) & \xrightarrow{k} & S^7 \end{array}$$

where A is an isometry. Since k is an embedding, this implies that M is a covering of $S^1(\sqrt{3}) \times S^2(\sqrt{3}/2)$ with covering map $k^{-1} \circ A \circ x$.

Remark. The condition that M is minimal may everywhere be replaced by the condition that the mean curvature vector H of M is parallel, since we have the following proposition [Y-K]$_2$.

Proposition 5.1. If M is an n-dimensional C-totally real submanifold of S^{2n+1}, then the following conditions are equivalent

(i) M is minimal

(ii) the mean curvature vector H of M is parallel.

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