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TAUBERIAN THEOREMS OF $J_p \to M_p$-TYPE

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Let $\{p_n\}$ be a sequence of non-negative numbers with $p_0 > 0$, $P_n := p_0 + \cdots + p_n$ for $n = 0, 1, \ldots$,

(1) $$p(x) := \sum_{n=0}^{\infty} p_n x^n \quad \text{for real } x$$

and, for a given sequence $\{s_n\}$ of complex numbers let

(2) $$p_s(x) := \sum_{n=0}^{\infty} p_n s_n x^n \quad \text{for real } x$$

and

$$t_n := \frac{1}{P_n} \sum_{k=0}^{n} p_k s_k \quad \text{for } n = 0, 1, \ldots .$$

We shall say that $\{s_n\}$ is M_p-limitable to σ, and write $M_p\text{-}\lim s_n = \sigma$, if $\lim_{n\to\infty} t_n = \sigma$. We shall say that $\{s_n\}$ is J_p-limitable to σ, and write $J_p\text{-}\lim s_n = \sigma$, if the series (1) has radius of convergence 1, the series (2) converges for $0 < x < 1$ and $\lim_{x\to 1-} p_s(x)/p(x) = \sigma$. It is known that both, the M_p-method and the J_p-method are regular if and only if $P_n \to \infty$. In this case $M_p\text{-}\lim s_n = \sigma$ implies $J_p\text{-}\lim s_n = \sigma$ (see Ishiguro [4]), but the converse is not always true. For many $\{p_n\}$ however $J_p\text{-}\lim s_n = \sigma$ implies $M_p\text{-}\lim s_n = \sigma$, if $\{s_n\}$ fulfills an additional condition, which we will call a Tauberian condition of $J_p \to M_p$-type.

If $p_n := 1$ for $n = 0, 1, \ldots$, then J_p is the Abel-method A_1 whilst M_p is the Cesàro-method C_1, and the following famous theorem of Hardy and Littlewood [3] (see [2, Theorem 94]) holds: $s_n = O_L(1)$ is a Tauberian condition of $A_1 \to C_1$-type. (For the definition of O_L, O and o see [2, p. 149].) But $s_n = O_L(1)$ is a Tauberian condition of $J_p \to M_p$-type for many other sequences $\{p_n\}$ (see, for example, [5, Theorem 4.1 and pp. 72/73] and [16]). Most of these results are special cases of

Theorem A ([16, Satz 4.3]). *Let* $P_n \to \infty$ *and*

(3) $$np_n = O(P_n).$$

Then $s_n = O_L(1)$ *is a Tauberian condition of* $J_p \to M_p$*-type.*

If $p_n := (n+1)^{-1}$ for $n = 0, 1, \ldots$, then J_p and M_p are the logarithmic methods L and l respectively, and the following theorem of Kokhanovskii [7, Theorem 1] holds : $s_n = O_L(\ln(n+1))$ is a Tauberian condition of $L \to l$-type. Now for $p_n := (n+1)^{-1}$ the condition $s_n = O_L(\ln(n+1))$ is equivalent to $np_n s_n = O_L(P_n)$, and this is a Tauberian condition of $J_p \to M_p$-type for many other sequences $\{p_n\}$ (see, for example, Kokhanovskii [8, Theorem 1] and Mikhalin [11, Theorem 5]). All these results are special cases of

Theorem B (Borwein [1, Theorem 2]). *Let* $P_n \to \infty$ *and*

$$np_n = o(P_n). \tag{4}$$

Then $np_n s_n = O_L(P_n)$ *is a Tauberian condition of* $J_p \to M_p$*-type.*

If (3) holds, then $np_n s_n = O_L(P_n)$ is implied by $s_n = O_L(1)$. So it is natural to ask, whether (4) in Theorem B can be replaced by (3). The answer is yes, and is a consequence of the following theorem because (3) is strictly stronger than

$$1 \leqq P_n/P_m \to 1 \text{ as } 1 < n/m \to 1 \ (m \to \infty). \tag{5}$$

Theorem 1. *Let* $P_n \to \infty$ *and* (5). *Then* $np_n s_n = O_L(P_n)$ *is a Tauberian condition of* $J_p \to M_p$*-type.*

For the proof of Theorem 1, which contains Theorems A and B as special cases, we need some lemmas.

Lemma 2. *Let* $P_n \to \infty$ *and* $P_{2n} = O(P_n)$. *Then*

a) $p(x)/p(x^2) = O(1)$ *as* $x \to 1-$.

b) $p(t^{1/n}) = O(P_n)$ *as* $n \to \infty$ *for every* $t \in (0, 1)$.

Lemma 2a) was proved by Kratz and Stadtmüller [9, Lemma 2, proof of (vii) $\Rightarrow$ (viii)]. To prove b) let $t \in (0, 1)$ be fixed. By a) there exists $K > 0$ such that $p(x) < Kp(x^2)$ for all $x \in (0, 1)$. Now we choose $m \in \mathbb{N}$ such that $\varepsilon := Kt^{m/2} < 1$. Then

$$\begin{aligned} p(t^{1/n}) &\leqq P_{mn} + \textstyle\sum_{k=mn+1}^{\infty} p_k t^{k/2n} t^{k/2n} \\ &\leqq P_{mn} + t^{m/2} p(t^{1/2n}) \leqq P_{mn} + Kt^{m/2} p(t^{1/n}), \end{aligned}$$

and therefore, $P_{2n} = O(P_n)$ implies

$$\frac{p(t^{1/n})}{P_n} \leqq \frac{P_{mn}}{P_n} \frac{1}{1-\varepsilon} = O(1) \text{ as } n \to \infty.$$

Lemma 3. *Let* $P_n \to \infty$, (5) *and* $np_n s_n = O_L(P_n)$. *Then*

$$p_s(x)/p(x) = O(1) \text{ as } x \to 1- \text{ implies } t_n = O(1).$$

Proof. We choose $K > 0$ with $np_n s_n > -KP_n$ for $n = 0, 1, \dots$ and define the non-negative sequence $\{\gamma_n\}$ by

$$\gamma_n := \begin{cases} |s_n| & \text{if } np_n = 0 \\ KP_n/np_n & \text{if } np_n \neq 0. \end{cases}$$

Then $s_n \geqq -\gamma_n$ for $n = 0, 1, \dots$ and $np_n\gamma_n = O(P_n)$. Now, following Borwein [1, proof of Theorem 2], we choose $t \in (0, 1)$ fixed and obtain

$$\frac{1}{p(t^{1/n})}\sum_{k=0}^{n} p_k s_k = O(1) \quad \text{as } n \to \infty.$$

From this, and because $P_{2n}/P_n = O(1)$ is a consequence of (5) (see [13]), $t_n = O(1)$ follows by Lemma 2b).

Special cases of Lemma 3 have been given by Kokhanovskii [7, Theorem 1] and [8, Lemma 1].

Lemma 4. *Let* $P_n \to \infty$, (5), $np_n s_n = O_L(P_n)$ *and* $t_n = O(1)$. *Then*

(6) $$\liminf (t_n - t_m) \geqq 0 \text{ as } Q_n/Q_m \to 1 \ (n > m \to \infty)$$

where $Q_n := P_0 + \cdots + P_n$ *for* $n = 0, 1, \dots$.

Proof. We have

$$t_n - t_{n-1} = \frac{p_n}{P_n}(s_n - t_{n-1}) \quad \text{for } n = 1, 2, \dots,$$

and therefore

$$t_n - t_m = \sum_{\nu=m+1}^{n} \frac{\nu p_\nu s_\nu}{P_\nu}\frac{1}{\nu} - \sum_{\nu=m+1}^{n} \frac{p_\nu t_{\nu-1}}{P_\nu} \quad \text{for } n > m > 0.$$

Hence, if $K > 0$ is a constant such that $np_n s_n \geqq -KP_n$ and $|t_n| \leqslant K$, we get

(7) $$t_n - t_m \geqq -K \ln\frac{n}{m} - K\left(\frac{P_n}{P_m} - 1\right) \quad \text{for } n > m > 0.$$

Now $Q_n/Q_m \to 1$ $(n > m \to \infty)$ implies $n/m \to 1$ and thus $P_n/P_m \to 1$ by

(5), so (6) follows from (7).

Special cases of Lemma 4 have been given by Kokhanovskii [7, Lemma 1] and [8, Lemma 3].

Lemma 5 ([15, Satz 3.9]). *Let* $P_n \to \infty$ *and* (5). *Then*

$$\liminf (s_n - s_m) \geqq 0 \quad as \quad P_n/P_m \to 1 \quad (n > m \to \infty) \tag{8}$$

is a Tauberian condition of $J_p \to c$*-type.*

Now we are able to give the

Proof of Theorem 1. Let $\{s_n\}$ be a sequence with $np_n s_n = O_L(P_n)$ and J_p-$\lim s_n = \sigma$. Then $t_n = O(1)$ by Lemma 3 and J_p-$\lim t_n = \sigma$. Therefore (6) holds by Lemma 4. Now (5) implies

$$1 \leqq Q_n/Q_m \to 1 \text{ as } 1 < n/m \to 1 \quad (m \to \infty)$$

(see [15, Nr. 5]), and so we get $\lim t_n = \sigma$, i.e. M_p-$\lim s_n = \sigma$, by Lemma 5.

Since $np_n s_n = O_L(P_n)$, the s_n in Theorem 1 have to be real. But clearly Theorem 1 holds for complex s_n, if we replace O_L by O. Thus we have the following corollary, which, for example, contains theorems of Kokhanovskii [6, Theorem 2], Teslenko [14, Theorem 2] and Mikhalin [10, Corollary 1] as special cases.

Corollary 6. *Let* $P_n \to \infty$ *and* (5). *Then* $np_n s_n = O(P_n)$ *is a Tauberian condition of* $J_p \to M_p$*-type.*

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