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THE DISTRIBUTION OF PYTHAGOREAN TRIPLES AND A THREE-DIMENSIONAL DIVISOR PROBLEM

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1. Introduction and statement of results. A triple of integers (r, s, n) is called Pythagorean if it satisfies $r^2+s^2=n^2$; for a large parameter x, let $A(x)$ denote the number of Pythagorean triples (r, s, n) with $1 \leqq n \leqq x$. Asymptotics for $A(x)$ have been established by Sierpinski [11], Fricker [4], [5] and Fischer [3]. The sharpest result to date is due to Stronina [12] and reads

$$A(x)=\frac{4}{\pi} x \log x+B x+O(x^{1/2} \exp (-c(\log x)^{3/5}(\log \log x)^{-1/5})) \quad (1)$$

with some $c>0$ and an explicitly given constant B. This estimate is based on Vinogradov's zero-free region of the Riemann zeta-function and the corresponding upper bound for the Dirichlet sum of the Moebius function; see Walfisz [14]. Furthermore, Stronina proved that the remainder term in (1) is not $o(x^{1/4})$.

The purpose of the present note is to give a conditional result somewhat sharper than (1), under the assumption of (a suitable extension of) Riemann's hypothesis.

Theorem 1. *Suppose that both Riemann's zeta-function $\zeta(s)$ and the Dirichlet L-function $L(s)$ corresponding to the non-principal character modulo 4 have no zero in the half-plane* $\operatorname{Re} s>1/2$. *Then, for any $\varepsilon>0$,*

$$A(x)=\frac{4}{\pi} x \log x+B x+O(x^{53/116+\varepsilon}). \quad (2)$$

Like most of the previous work, our argument is based on the factorization of the Dirichlet series

$$\sum_{n=1}^{\infty} r(n^2) n^{-s}=4(\zeta(s))^2(\zeta(2 s))^{-1} L(s)(1+2^{-s})^{-1} \quad (\operatorname{Re} s>1). \quad (3)$$

(Here $r(m)$ is the standard notation for the number of integer pairs (p, q) with $p^2+q^2=m$.) This suggests to consider $r(n^2)$ as a kind of convolution of the Moebius function μ with a certain three-dimensional divisor function.

To deal with the latter, we establish a result on Piltz' divisor problem in residue classes (for dimension 3) which is "unconditional" and perhaps of some interest of its own.

Theorem 2. *For $j=1,2,3$, let q_j and m_j be fixed natural numbers satisfying $1 \leqq q_j \leqq m_j$. For a positive integer n, let $d_3^*(n) = d_3^*(n;\ q_1, m_1;\ q_2, m_2;\ q_3, m_3)$ denote the number of triples (n_1, n_2, n_3) of positive integers for which $n_1 n_2 n_3 = n$ and $n_j \equiv q_j (\operatorname{mod} m_j)$, $j=1,2,3$. Then, for a large parameter y,*

$$\begin{aligned} D_3^*(y) &:= \textstyle\sum_{n \leq y} d_3^*(n;\ q_1, m_1;\ q_2, m_2;\ q_3, m_3) \\ &= yP_2(\log y) + O(y^{43/96+\varepsilon}) \end{aligned} \tag{4}$$

(for any $\varepsilon > 0$) where P_2 is a quadratic polynomial and, in fact,

$$yP_2(\log y) = \operatorname{Res}_{s=1}\left((m_1 m_2 m_3)^{-s} \zeta\left(s, \frac{q_1}{m_1}\right) \zeta\left(s, \frac{q_2}{m_2}\right) \zeta\left(s, \frac{q_3}{m_3}\right) y^s s^{-1}\right), \tag{4'}$$

where $\zeta(s, a)$ is Hurwitz' zeta-function.

Remark. This result is a straightforward generalization of Kolesnik's work [6] dealing with the special case $m_1 = m_2 = m_3 = 1$. In section 2, we give a draft of a proof, indicating which modifications are necessary. In section 3 we then apply a method of Montgomery and Vaughan [8] (instead of some elementary convolution argument) to derive theorem 1 from theorem 2.

In section 4, we investigate the distribution of Pythagorean triples in a different direction, employing a method of Pintz [9] to obtain a certain strengthening of the above-mentioned lower estimate for the error term due to Stronina.

Theorem 3. *For $x \geqq 1$, define*

$$E(x) := A(x) - \frac{4}{\pi} x \log x - Bx.$$

Then there exists a positive constant c_1 such that, for X sufficiently large,

$$\int_1^X |E(x)|\, dx \geqq c_1 X^{5/4}.$$

2. Sketch of proof of theorem 2. For the first part, we follow the argument of Atkinson [2]. The generating function of $d_3^*(n)$ is clearly given by

$$Z(s) := \prod_{j=1}^{3} m_j^{-s}\zeta(s, q_j m_j^{-1}) = \sum_{n=1}^{\infty} d_3^*(n) n^{-s} \quad (\mathrm{Re}\, s > 1). \tag{5}$$

Let y be half an odd integer and $T < y^{53/96} < 2T$, then the truncated Perron's formula (e.g. Prachar [10], p. 376) yields for $\varepsilon > 0$

$$D_3^*(y) = (2\pi i)^{-1} \int_{1+\varepsilon-iT}^{1+\varepsilon+iT} Z(s) y^s s^{-1} ds + O(y^{1+\varepsilon} T^{-1}). \tag{6}$$

By Hurwitz' formula (see Apostol [1], p. 257) we have for Re $s > 1$

$$\zeta(1-s, a) = (2\pi)^{-s}\Gamma(s) 2\sum_{n=1}^{\infty} n^{-s} \cos\left(2\pi n a - \frac{\pi}{2} s\right). \tag{7}$$

Using Stirling's formula for $\Gamma(s)$ (e.g. Landau [7], p. 227), we get

$$\zeta(-\varepsilon + it, a) \ll |\Gamma(1+\varepsilon-it)| \exp\left(\frac{\pi}{2}|t|\right) \ll (1+|t|)^{1/2+\varepsilon}$$

and thus, by (5) and the Phragmén-Lindelöf principle ([7], p. 229),

$$Z(\sigma+it) \ll (1+|t|)^{(3/2)(1+\varepsilon-\sigma)} \tag{8}$$

uniformly in the strip $-\varepsilon \leqq \sigma \leqq 1+\varepsilon$. Therefore, we may shift the path of integration in (6) to the line segment from $-\varepsilon - iT$ to $-\varepsilon + iT$, noting that the integrand has a pole of order 3 at $s = 1$ with residue $yP_2(\log y)$ and a simple pole at $s = 0$ with residue $Z(0)$. Defining

$$\Delta_3^*(y) := D_3^*(y) - yP_2(\log y),$$

we obtain, estimating the remainder integrals by (8),

$$\Delta_3^*(y) = (2\pi i)^{-1} \int_{-\varepsilon-iT}^{-\varepsilon+iT} Z(s) y^s s^{-1} ds + O(y^{1+\varepsilon} T^{-1}).$$

Now let $\mathscr{S} = \{-1, 1\}^3$, and define for $b = (b_1, b_2, b_3) \in \mathscr{S}$, $(n_1, n_2, n_3) \in \mathbb{N}^3$,

$$\beta_b(n_1, n_2, n_3) := 2\pi \sum_{j=1}^{3} b_j n_j q_j m_j^{-1}, \quad \gamma_b = \sum_{j=1}^{3} b_j,$$
$$\lambda_{n,b} := \sum_{n_1 n_2 n_3 = n} \cos\beta_b(n_1, n_2, n_3), \quad \mu_{n,b} := \sum_{n_1 n_2 n_3 = n} \sin\beta_b(n_1, n_2, n_3).$$

Then it follows from (5) and (7), by a short computation, that, for Re $s > 1$,

$$Z(1-s) = \sum_{b\in\mathscr{S}} Z_b(1-s)$$

where

$$Z_b(1-s):=(m_1m_2m_3)^{s-1}(8\pi^3)^{-s}\Gamma^3(s)\sum_{n=1}^{\infty}n^{-s}\left(\lambda_{n,b}\cos\left(\frac{\pi}{2}\gamma_b s\right)+\right.$$
$$\left.+\mu_{n,b}\sin\left(\frac{\pi}{2}\gamma_b s\right)\right). \tag{10}$$

By Stirling's formula (with $s=\sigma+it$ as usual),

$$(1-s)^{-1}\Gamma^3(s)=-2\pi\sqrt{3}\,3^{2-3s}\Gamma(3s-2)(1+O((|t|+1)^{-1})) \tag{11}$$

uniformly in any strip $\sigma_1\leqq\sigma\leqq\sigma_2$. We make the change of variable $s\to 1-s$ in (9) and define $z_n=8\pi^3ny(m_1m_2m_3)^{-1}$ for short. Thus

$$\Delta_3^*(y)=\sum_{b\in\mathscr{S}}(2\pi i)^{-1}\int_{1+\varepsilon-iT}^{1+\varepsilon+iT}Z_b(1-s)y^{1-s}(1-s)^{-1}ds+O(y^{1+\varepsilon}T^{-1}) \tag{12}$$

and (for fixed $b\in\mathscr{S}$) this integral equals, by (10) and (11),

$$-2\pi\sqrt{3}\,y(m_1m_2m_3)^{-1}\sum_{n=1}^{\infty}\int_{1+\varepsilon-iT}^{1+\varepsilon+iT}z_n^{-s}3^{2-3s}\Gamma(3s-2)\left(\lambda_{n,b}\cos\left(\frac{3\pi}{2}gs\right)+\right.$$
$$\left.+\mu_{n,b}\sin\left(\frac{3\pi}{2}gs\right)\right)(\delta_{3,|\gamma_b|}+O(|t|^{-1}))ds \tag{13}$$

where δ is Kronecker's symbol and $g=\gamma_b|\gamma_b|^{-1}$. Since, again by Stirling's formula, for $s=1+\varepsilon+it$

$$3^{-3s}\Gamma(3s-2)\exp\left(\pm\frac{3\pi}{2}is\right)|t|^{-1}=O(|t|^{-1/2+3\varepsilon}),$$

the contribution of the order term in (13) is

$$\ll y\sum_{n=1}^{\infty}z_n^{-1-\varepsilon}(|\lambda_{n,b}|+|\mu_{n,b}|)T^{1/2+3\varepsilon}\ll yT^{-1}$$

(by choice of T). To deal with the main term in (13) (which only occurs for $b=\pm(1,1,1)$), we make the change of variable $3s-2\to s$ and obtain

$$\Delta_3^*(y)=\frac{1}{\pi\sqrt{3}}y^{1/3}(m_1m_2m_3)^{-1/3}\sum_{n=1}^{\infty}n^{-2/3}(\lambda_nI_n^{(1)}+\mu_nI_n^{(2)})+$$
$$+O(y^{1+\varepsilon}T^{-1}) \tag{14}$$

where

$$I_n^{(1)}:=(2\pi i)^{-1}\int_{1+3\varepsilon-iT}^{1+3\varepsilon+iT}z_n^{-s/3}3^{-s}\Gamma(s)\cos\left(\frac{\pi}{2}s\right)ds$$

and $I_n^{(2)}$ is defined by replacing cos by sin ; λ_n, μ_n are the values of $\lambda_{n,b}$, $\mu_{n,b}$ for $b=(1,1,1)$. We now observe that the evaluation of $I_n^{(1)}$ and $I_n^{(2)}$ is contained in the analysis of Atkinson [2], lemma 1 and 2. Thus we arrive at

$$\Delta_3^*(y)=\frac{1}{\pi\sqrt{3}}y^{1/3}(m_1m_2m_3)^{-1/3}\sum_{n_1n_2n_3\le N}\cos\left(6\pi\left(\frac{yn_1n_2n_3}{m_1m_2m_3}\right)^{1/3}-\right.$$
$$\left.-2\pi\sum_{j=1}^{3}\frac{q_j}{m_j}n_j\right)(n_1n_2n_3)^{-2/3}+O(y^{1+\varepsilon}T^{-1}) \qquad (15)$$

where N is an integer such that $N+1/2=(m_1m_2m_3T)^3(8\pi^3y)^{-1}$. For $m_1=m_2=m_3=1$, this coincides with Atkinson's formula (5.1) which was used by Kolesnik to obtain his result in [6]. But it is easily verified that Kolesnik's argument remains unaffected by the additional linear terms $(q_j/m_j)n_j$ and yields the estimate $O(y^{11/96+\varepsilon})$ for the trigonometric sum above. This establishes theorem 2.

3. Proof of theorem 1. We may rewrite (3) as

$$F(s):=\sum_{n=1}^{\infty}r(n^2)n^{-s}=4\zeta_0(s)(\zeta_0(2s))^{-1}\zeta(s)L(s) \qquad (16)$$

where

$$\zeta_0(s):=(1-2^{-s})\zeta(s)=\sum_{u=1}^{\infty}u^{-s},$$

u and v denoting odd integers throughout the sequel. Thus

$$\frac{1}{4}A(x)=\frac{1}{4}\sum_{n\le x}r(n^2)=\sum_{uv^2km\le x}\mu(v)\chi(m)$$
$$=\sum_{v\le y}+\sum_{v>y}=:S_1+S_2 \qquad (17)$$

where χ denotes the non-principal character modulo 4 and $y<\sqrt{x}$ is a parameter remaining at our disposition. By theorem 2,

$$S_1=\sum_{v\le y}\mu(v)\sum_{ukm\le xv^{-2}}\chi(m)$$
$$=\sum_{v\le y}\mu(v)\sum_{n\le xv^{-2}}(d_3^*(n;\,1,2;\,1,1;\,1,4)-$$
$$-d_3^*(n;\,1,2;\,1,1;\,3,4))$$
$$=\sum_{v\le y}\mu(v)(R(xv^{-2})+O((xv^{-2})^{43/96+\varepsilon}))$$

where $R(w)$ is the residue of $w^ss^{-1}\zeta_0(s)\zeta(s)L(s)$ at $s=1$. Consequently,

$$S_1=\mathrm{Res}_{s=1}(x^ss^{-1}f_1(s)\zeta_0(s)\zeta(s)L(s))+O(x^{43/96+\varepsilon}y^{5/48}) \qquad (18)$$

where

$$f_1(s) := \sum_{v \leq y} \mu(v) v^{-2s}.$$

To deal with S_2, we define

$$f_2(s) := (\zeta_0(2s))^{-1} - f_1(s)$$

and conclude by the argument of Titchmarsh [13], p. 315, that, under Riemann's hypothesis,

$$f_2(s) = O(y^{1/2-2\sigma+\varepsilon}(|t|^{\varepsilon}+1)) \tag{19}$$

uniformly in $\sigma \geqq 1/4+\varepsilon$, for any $\varepsilon > 0$ (see also Montgomery and Vaughan [8], p. 250). By the truncated Perron's formula (Prachar [10], p. 376), for any $U > 0$,

$$S_2 = (2\pi i)^{-1} \int_{2-iU}^{2+iU} f_2(s)\zeta_0(s)\zeta(s)L(s)x^s s^{-1} \mathrm{d}s + O(x^2 U^{-1}). \tag{20}$$

According to [13], p. 283, our assumptions on $\zeta(s)$ and $L(s)$ imply the validity of the Lindelöf hypothesis for these functions, thus, together with (19),

$$f_2(s)\zeta_0(s)\zeta(s)L(s) = O(y^{1/2-2\sigma+\varepsilon}(|t|^{\varepsilon}+1)) \tag{21}$$

uniformly in $\sigma \geqq 1/2+\varepsilon$, for any $\varepsilon > 0$. We now shift the path of integration in (20) to the line segment from $1/2+\varepsilon-iU$ to $1/2+\varepsilon+iU$, estimate the remainder integrals by (21) and choose $U = x^3$ to obtain (with a new $\varepsilon > 0$)

$$S_2 = \mathrm{Res}_{s=1}(x^s s^{-1} f_2(s)\zeta_0(s)\zeta(s)L(s)) + O(x^{1/2+\varepsilon} y^{-1/2}). \tag{22}$$

Finally we combine the results (18) and (22), choose $y = x^{5/58}$ and infer that

$$A(x) = \mathrm{Res}_{s=1}(x^s s^{-1} F(s)) + O(x^{53/116+\varepsilon}). \tag{23}$$

Since this residue is easily computed to $(4/\pi)x \log x + Bx$ (cf. (16)), the proof of theorem 1 is thereby complete.

4. Proof of theorem 3. We follow the argument of Pintz [9]. By the definitions of $E(x)$ and $F(s)$,

$$\begin{aligned}\int_1^{\infty} E(x) x^{-s-1} \mathrm{d}x &= \frac{1}{s} F(s) - \frac{4}{\pi}(s-1)^{-2} - B(s-1)^{-1} =: H(s) =: \\ &=: \frac{N(s)}{(2s-1)s(s-1)^2\zeta(2s)(1+2^{-s})}\end{aligned} \tag{24}$$

for $\operatorname{Re} s > 1$, where $N(s)$ is an entire function. Let $\rho = 1/2 + i\gamma$ be some fixed simple zero of the Riemann zeta-function (with minimal $|\gamma|$, say) such that $\zeta(\rho/2)L(\rho/2) \neq 0$, then we put

$$g(s) := s(s-1)^2(2s-1)\zeta(2s)(1+2^{-s})\left(s-\frac{\rho}{2}\right)^{-1}(s+2)^{-c} \tag{25}$$

where c is a sufficiently large positive constant. Thus, in $-1 \leqq \operatorname{Re} s \leqq 2$, $H(s)g(s)$ is meromorphic with the only pole (of order 1) at $s = \rho/2$ (since $N(\rho/2) \neq 0$). Furthermore, it follows (from well-known order results for the zeta- and L-functions, e.g. [1]) that

$$\int_{\sigma-i\infty}^{\sigma+i\infty} |g(s)||H(s)|^j \mathrm{d}s < \infty \tag{26}$$

for $\sigma = -1$ or 2 and $j = 0$ or 1. Moreover, we define for $a > 0$

$$w(a) := \int_{2-i\infty}^{2+i\infty} g(s)a^{s+1}\mathrm{d}s \tag{27}$$

and conclude, by shifting the line of integration either to $\sigma = -1$ or to $\sigma \to \infty$ (noting that $g(s)$ is regular in $\operatorname{Re} s > -2$), that $w(a) = 0$ for $0 < a < 1$ and $w(a) = O(1)$ for $a \geqq 1$. We finally put

$$U(X) := X^{-1}\int_1^X E(x)w(Xx^{-1})\mathrm{d}x \tag{28}$$

and infer, by (27) and the dominated convergence theorem, that

$$U(X) = X^{-1}\int_1^\infty E(x)w(Xx^{-1})\mathrm{d}x = \int_{2-i\infty}^{2-i\infty} g(s)H(s)X^s\mathrm{d}s. \tag{29}$$

Shifting the line of integration to $\sigma = -1$, we obtain

$$\begin{aligned} U(X) &= 2\pi i \operatorname{Res}_{s=\rho/2}(g(s)H(s)X^s) + \int_{-1-i\infty}^{-1+i\infty} g(s)H(s)X^s\mathrm{d}s \\ &= CX^{\rho/2} + O(X^{-1}) \end{aligned} \tag{30}$$

where $C \neq 0$ is a complex constant. On the other hand, by (28) and the boundedness of $w(a)$,

$$\int_1^X |E(x)|\mathrm{d}x \gg X|U(X)| \gg X|X^{\rho/2}| = X^{5/4} \tag{31}$$

which completes the proof of theorem 3.

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Added in proof. The first named author has meanwhile established the result of Theorem 1 under the Riemann Hypothesis alone (without any unproven assumption about the L-series involved): See *Monatsh. f. Math.* **106** (1988), 57-63.