Math. J. Okayama Univ. 39 (1989), 207-212

ASYMPTOTIC BEHAVIOR OF A CERTAIN MULTIPLICATIVE FUNCTION

ISAO KIUCHI

1. Statement of the result. Let $-1/2 < a \leqq 0$, n be a positive integer and $\sigma_a(n) = \sum_{d|n} d^a$ be the sum of the a-th powers of positive divisors of n, so that $\sigma_0(n) = d(n)$, the divisor function. Let $s = \sigma + it$ be a complex variable, and

$$E_a(s\,;\,h/k) = \sum_{n=1}^{\infty} \frac{\sigma_a(n)e(hn/k)}{n^s}$$

where h and k are co-prime integers with $1 \leqq k$, $e(t) = e^{2\pi it}$ and $Re(s) > 1$. The function $E_a(s\,;h/k)$ can be analytically continued to a meromorphic function in the whole complex plane (see [1]). We note $E_0(s\,;\,h/1)$ is equal to square of the Riemann zeta function $\zeta(s)$.

Titchmarsh [2] proved an approximate functional equation for $\zeta^2(s)$. His proof starts from the following identity, valid for $\sigma > -1/4$:

$$(1.1)\quad \zeta^2(s) = \sum_{n\leq x} \frac{d(n)}{n^s} - x^{-s} \sum_{n\leq x} d(n) + \frac{2s-s^2}{(s-1)^2} x^{1-s} + \frac{1}{4} x^{-s}$$
$$+ \frac{s}{s-1}(2\gamma + \log x) x^{1-s}$$
$$-2^{4s}\pi^{2s-2} s \sum_{n=1}^{\infty} \frac{d(n)}{n^{1-s}} \int_{4\pi\sqrt{nx}}^{\infty} \frac{K_1(\nu) + (\pi/2) Y_1(\nu)}{\nu^{2s}} d\nu$$

where x is a positive number, $K_1(\nu)$ and $Y_1(\nu)$ are Bessel functions, γ is an Euler's constant.

The purpose of this paper is to derive a formula of the type (1.1) for the function $E_a(s\,;\,h/k)$. In the derivation of (1.1), Titchmarsh used Fourier integrals for Bessel functions and the functional equation of $\zeta(s)$. The method of Titchmarsh's proof of the above formula (1.1) can be modified so as to give an analogous expression for $E_a(s\,;\,h/k)$, and such a modification gives the following

Theorem. *If* $\sigma > -1/4 + a/2$, *we have, for* $-1/2 < a < 0$

(1.2) $E_a(s\,;\,h/k)$

$$= \sum_{n\le x}\frac{\sigma_a(n)e(hn/k)}{n^s} - x^{-s}\sum_{n\le x}\sigma_a(n)e(hn/k) + E_a(0\,;\,h/k)x^{-s}$$

$$+\frac{s}{s-1}k^{a-1}\zeta(1-a)x^{1-s}+\frac{s}{(s-a-1)(1+a)}k^{-1-a}\zeta(1+a)x^{1+a-s}$$

$$-2^{4s-2a}\pi^{2s-a-2}k^{1+a-2s}s\cos\left(\frac{\pi}{2}a\right)\sum_{n=1}^{\infty}\frac{\sigma_a(n)e(-\bar{h}n/k)}{n^{1+a-s}}\cdot$$

$$\int_{4\pi\sqrt{nx}/k}^{\infty}\frac{K_{1+a}(v)+(\pi/2)Y_{1+a}(v)}{v^{2s-a}}dv$$

$$-2^{4s-2a-1}\pi^{2s-a-1}k^{1+a-2s}s\sin\left(\frac{\pi}{2}a\right)\sum_{n=1}^{\infty}\frac{\sigma_a(n)e(-\bar{h}n/k)}{n^{1+a-s}}\cdot$$

$$\int_{4\pi\sqrt{nx}/k}^{\infty}\frac{J_{1+a}(v)}{v^{2s-a}}dv$$

$$-i2^{4s-2a+1}\pi^{2s-a-2}k^{1+a-2s}s\cos\left(\frac{\pi}{2}a\right)\sum_{n=1}^{\infty}\frac{\sigma_a(n)\sin(2\pi\bar{h}n/k)}{n^{1+a-s}}\cdot$$

$$\int_{4\pi\sqrt{nx}/k}^{\infty}\frac{K_{1+a}(v)}{v^{2s-a}}dv,$$

and for $a=0$

(1.3) $E_0(s\,;\,h/k)$

$$=\sum_{n\le x}\frac{d(n)e(hn/k)}{n^s}-x^{-s}\sum_{n\le x}d(n)e(hn/k)+E_0(0\,;\,h/k)x^{-s}$$

$$+\frac{s}{s-1}k^{-1}(2\gamma+\log x-2\log k)x^{1-s}+\frac{2s-s^2}{(s-1)^2}k^{-1}x^{1-s}$$

$$-2^{4s}\pi^{2s-2}k^{1-2s}s\sum_{n=1}^{\infty}\frac{d(n)e(-\bar{h}n/k)}{n^{1-s}}\int_{4\pi\sqrt{nx}/k}^{\infty}\frac{K_1(v)+(\pi/2)Y_1(v)}{v^{2s}}dv$$

$$-i2^{4s+1}\pi^{2s-2}k^{1-2s}s\sum_{n=1}^{\infty}\frac{d(n)\sin(2\pi\bar{h}n/k)}{n^{1-s}}\int_{4\pi\sqrt{nx}/k}^{\infty}\frac{K_1(v)}{v^{2s}}dv,$$

where $K_{1+a}(v)$, $J_{1+a}(v)$ *and* $Y_{1+a}(v)$ *are Bessel functions, and the class* $\bar{h}$ *(mod* k*) is defined by* $h\bar{h}\equiv 1$ *(mod* k*).*

Remark 1. As an immediate corollary of (1.3) with $k=1$, we get (1.1).

Remark 2. Further investigations of the asymptotic behavior of $E_a(s\,;\,h/k)$ will appear in forthcoming papers.

2. Lemma. *The function $E_a(s\,;h/k)$ can be analytically continued to a meromorphic function, which is regular in the whole complex plane up to two simple poles at $s=1,\ 1+a\ (-1<a<0)$, and is regular in the whole complex plane up to a double pole at $s=1\ (a=0)$. The function $E_a(s\,;h/k)$ satisfies the functional equation*

$$(2.1)\quad E_a(s\,;h/k) = \frac{1}{\pi}\left(\frac{k}{2\pi}\right)^{1+a-2s}\Gamma(1-s)\Gamma(1+a-s)\left\{\cos\left(\frac{\pi}{2}a\right)E_a(1+a-s\,;\bar{h}/k) - \cos\left(\pi s-\frac{\pi}{2}a\right)E_a(1+a-s\,;-\bar{h}/k)\right\},$$

and has for $-1<a<0$ the Laurent expansions

$$(2.2)\quad E_a(s\,;h/k) = \begin{cases} \dfrac{k^{-1+a}\zeta(1-a)}{s-1}+\cdots\cdots & \text{at } s=1 \\ \dfrac{k^{-1-a}\zeta(1+a)}{s-a-1}+\cdots\cdots & \text{at } s=1+a \end{cases}$$

and has for $a=0$ the Laurent expansion

$$(2.3)\quad E_0(s\,;h/k) = \frac{k^{-1}}{(s-1)^2}+\frac{k^{-1}(2\gamma-2\log k)}{s-1}+\cdots\cdots \qquad \text{at } s=1.$$

This lemma has been proved in Kiuchi[1].

3. Proof of Theorem. Suppose firstly that $\sigma>1$, and we shall calculate the integral

$$(3.1)\quad I=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{sx^{w-s}}{w(s-w)}E_a(w\,;h/k)dw \qquad (1<c<\sigma)$$

in two ways. We substitute the definition of the series $E_a(w\,;h/k)$ to the right-hand side of (3.1), and carry out the termwise integration. By the theorem of residues, we obtain

$$(3.2)\quad I=E_a(s\,;h/k)-\sum_{n\le x}\frac{\sigma_a(n)e(hn/k)}{n^s}+x^{-s}\sum_{n\le x}\sigma_a(n)e(hn/k).$$

On the other hand, moving the contour in I to $Re(w)=a-\varepsilon\ (\varepsilon>0)$, we have

$$I=\frac{1}{2\pi i}\int_{-\varepsilon-i\infty}^{-\varepsilon+i\infty}\frac{sx^{w-s}}{w(s-w)}E_0(w\,;\,h/k)\mathrm{d}w+E_0(0\,;\,h/k)x^{-s}$$
$$+\frac{s}{s-1}k^{-1}(2\gamma+\log x-2\log k)x^{1-s}+\frac{2s-s^2}{(s-1)^2}k^{-1}x^{1-s}$$

for $a=0$, and

$$I=\frac{1}{2\pi i}\int_{a-\varepsilon-i\infty}^{a-\varepsilon+i\infty}\frac{sx^{w-s}}{w(s-w)}E_a(w\,;\,h/k)\mathrm{d}w+E_a(0\,;\,h/k)x^{-s}$$
$$+\frac{s}{(s-1-a)(1+a)}k^{-1-a}\zeta(1+a)x^{1+a-s}+\frac{s}{s-1}k^{a-1}\zeta(1-a)x^{1-s}$$

for $-1/2<a<0$.

Next, by the functional equation (2.1), the first term of the right-hand side of the above two identity is represented by

$$\frac{1}{2\pi i}\int_{a-\varepsilon-i\infty}^{a-\varepsilon+i\infty}\frac{sx^{w-s}}{w(s-w)}E_a(w\,;\,h/k)\mathrm{d}w$$
$$=2^{2+a}\pi^a k^{-1-a}\cos\left(\frac{\pi}{2}a\right)\sum_{n=1}^{\infty}\sigma_a(n)e(-\bar{h}n/k)\frac{1}{2\pi i}\cdot$$
$$\int_{1+\varepsilon-i\infty}^{1+\varepsilon+i\infty}\frac{sx^{1+a-s-u}}{(1+a-u)(s-1-a+u)}\cdot$$
$$\Gamma(u)\Gamma(u-a)\cos^2\left(\frac{\pi}{2}u-\frac{\pi}{2}a\right)\left(\frac{2\pi\sqrt{n}}{k}\right)^{-2u}du$$
$$-2^{1+a}\pi^a k^{-1-a}\sin\left(\frac{\pi}{2}a\right)\sum_{n=1}^{\infty}\sigma_a(n)e(-\bar{h}n/k)\frac{1}{2\pi i}\cdot$$
$$\int_{1+\varepsilon-i\infty}^{1+\varepsilon+i\infty}\frac{sx^{1+a-s-u}}{(1+a-u)(s-1-a+u)}\cdot$$
$$\Gamma(u)\Gamma(u-a)\sin(\pi u-\pi a)\left(\frac{2\pi\sqrt{n}}{k}\right)^{-2u}du$$
$$+i2^{2+a}\pi^a k^{-1-a}\cos\left(\frac{\pi}{2}a\right)\sum_{n=1}^{\infty}\sigma_a(n)\sin(2\pi\bar{h}n/k)\frac{1}{2\pi i}\cdot$$
$$\int_{1+\varepsilon-i\infty}^{1+\varepsilon+i\infty}\frac{sx^{1+a-s-u}}{(1+a-u)(s-1-a+u)}\cdot$$
$$\Gamma(u)\Gamma(u-a)\left(\frac{2\pi\sqrt{n}}{k}\right)^{-2u}du$$
$$=F_1-F_2+F_3,\ \text{say.}$$

Applying Mellin's inversion formula (see e.g. [3]), we obtain that if $-1/2<a\leqq 0$ and $1+a<b<5/2+a$, then

$$\frac{K_{1+a}(\nu)+(\pi/2)Y_{1+a}(\nu)}{\nu^{1+a}}$$
$$=\frac{1}{2\pi i}\int_{b/2-i\infty}^{b/2+i\infty}2^{2u-a-1}\Gamma(u)\Gamma(u-a-1)\cos^2\left(\frac{\pi}{2}u-\frac{\pi}{2}a\right)\nu^{-2u}du,$$
$$\frac{J_{1+a}(\nu)}{\nu^{1+a}}=\frac{1}{2\pi i}\int_{b/2-i\infty}^{b/2+i\infty}2^{2u-a-1}\frac{\Gamma(u)}{\Gamma(2+a-u)}\nu^{-2u}du,$$
$$\frac{K_{1+a}(\nu)}{\nu^{1+a}}=\frac{1}{2\pi i}\int_{b/2-i\infty}^{b/2+i\infty}2^{2u-a-2}\Gamma(u)\Gamma(u-a-1)\nu^{-2u}du.$$

Multiplying these by $\nu^{1+2a-2s}$ and integrating over the interval $[4\pi\sqrt{nx}/k, \infty)$, we obtain

$$F_1=-2^{4s-2a}\pi^{2s-2-a}k^{1+a-2s}s\cos\left(\frac{\pi}{2}a\right)\sum_{n=1}^{\infty}\frac{\sigma_a(n)e(-\bar{h}n/k)}{n^{1+a-s}}\cdot\int_{4\pi\sqrt{nx}/k}^{\infty}\frac{K_{1+a}(\nu)+(\pi/2)Y_{1+a}(\nu)}{\nu^{2s-a}}d\nu,$$
$$F_2=2^{4s-1-2a}\pi^{2s-1-a}k^{1+a-2s}s\sin\left(\frac{\pi}{2}a\right)\sum_{n=1}^{\infty}\frac{\sigma_a(n)e(-\bar{h}n/k)}{n^{1+a-s}}\cdot\int_{4\pi\sqrt{nx}/k}^{\infty}\frac{J_{1+a}(\nu)}{\nu^{2s-a}}d\nu,\text{ and}$$
$$F_3=-i2^{4s-2a+1}\pi^{2s-2a}k^{1+a-2s}s\cos\left(\frac{\pi}{2}a\right)\sum_{n=1}^{\infty}\frac{\sigma_a(n)\sin(2\pi\bar{h}n/k)}{n^{1+a-s}}\cdot\int_{4\pi\sqrt{nx}/k}^{\infty}\frac{K_{1+a}(\nu)}{\nu^{2s-a}}d\nu.$$

Here we note that the order of the integrations can be inverted in view of the absolute convergence of the double integral. Hence, together with (3.2) this gives the formula (1.2) and (1.3), in the first place for $\sigma>1$, and by analytic continuation in the half-plane $\sigma>-1/4+a/2$. This completes the proof of Theorem.

REFERENCES

[1] I. KIUCHI : On an exponential sum involving the arithmetic function $\sigma_a(n)$, Math. J. Okayama Univ. 29 (1987), 193-205.

[2] E. C. TITCHMARSH : The approximate functional equation for $\zeta^2(s)$, Quart. J. Math. Oxford 9 (1937), 109-114.

[3] E. C. TITCHMARSH : Introduction to the theory of Fourier integrals, Oxford Univ. Press (Clarendon), London, New York 1948.

KUGENUMA-ISHIGAMI 1-13-13-905
FUJISAWA, KANAGAWA, 251 JAPAN

(*Received June 27, 1988*)