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THE A-COHOMOLOGY OF THE STUNTED QUATERNIONIC (QUASI-)PROJECTIVE SPACES

Dedicated to Professor Masahiro Sugawara on his 60th birthday

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Let KO be the real K-spectrum, and kO and $kSpin$ be the -1 and 3 connected covering spectra of KO respectively. Then the stable Adams operation $\psi^3 : KO_{(2)} \to KO_{(2)}$ is defined, and $\psi^3-1 : KO_{(2)} \to KO_{(2)}$ lifts to the operation $\psi: kO_{(2)} \to kSpin_{(2)}$, where $E_{(2)}$ denotes the spectrum E localized at 2. We define a spectrum A to be the fiber spectrum of ψ. Then A is a connected spectrum, and its coefficient groups are isomorphic to the groups consisting of J-classes and μ classes in the stable homotopy groups of the spheres. Thus the spectrum A represents the $\mathrm{Im}\,J$ theory exploited by several authors (cf. [11], [10], [4]). In this note we calculate the A-cohomology groups of the stunted quaternionic (quasi-) projective spaces. These groups are closely related to the KO-theory quaternionic James numbers, and the original idea for utility of the A-theory is due to the work of K. Knapp [9].

Let HP^n be the quaternionic projective space, and ξ denote the canonical quaternionic line bundle over HP^n. ζ denotes the 3 dimensional real vector bundle over HP^{n-1} associated with the adjoint representation of S^3. Then the quaternionic quasi-projective space Q_n is defined to be the Thom space $(HP^{n-1})^{\zeta}$. Our object is the calculation of the cohomology groups $A^*(HP^n/HP^{m-1})$ and $A^*(Q_n/Q_{m-1})$. To treat these two cases, we denote by KP^n one of the spaces HP^n and ΣQ_n, where the latter space is a reduced suspension which we use for the sake of adjusting the dimension, that is, $\dim KP^n = 4n$ in both cases. We put $KP^n_m = KP^n/KP^{m-1}$ for the stunted projective space. Throughout the paper, the integers n and m used in the index of KP^n_m are always assumed to satisfy $1 \leqq m \leqq n$.

This paper is organized as follows : In §1, we prepare the notations for the KO-theory of the projective spaces. In §2 we discuss the free part of $A^i(KP^n_m)$, and §3 is devoted to the calculation of the stable Adams operations. In §4, we describe the cohomology groups $A^i(KP^n_m)$.

1. Preliminaries. The cohomology groups appeared in this paper are the K-cohomologies $KO^*(X)$, $kO^*(X)$ and $kSpin^*(X)$, A-cohomology $A^*(X)$ and the ordinary cohomologies $H^*(X;F)$ for $F=Z$(integers) or Q(rationals). Throughout the paper, these cohomology groups are always assumed to be the reduced cohomology groups. $Z_{(2)}$ and $Z/2$ denote the groups of the localized integers at 2 and the mod 2 integers respectively, and we denote by $G_{(2)}$ the group $G\otimes Z_{(2)}$ for an abelian group G.

For $i\in Z$, let $g_i\in KO^0(S^{4i})\cong Z$ and $\alpha\in KO^0(S^1)\cong Z/2$ be the respective generators. Then $KO^*(S^0)\cong Z[\alpha, g_1, g_2, g_2^{-1}]/(2\alpha, \alpha^3, \alpha g_1, g_1^2-4g_2)$, and thus we have $g_{2i}=g_2^i$, $g_{2i+1}=g_1g_2^i$ and $\alpha g_{2i+1}=0$ for $i\in Z$. Let $a(i)=1$ for an even integer i and $a(i)=2$ for an odd integer i, and let $\binom{i}{j}$ be the binomial coefficient.

For the canonical line bundle ξ over HP^n, let $X=[\xi-1]\in KO^4(HP^n)$ and $x\in H^4(HP^n;Z)$ denote the respective Euler classes. Then we have a $KO^*(S^0)$-algebra isomorphism $KO^*(HP_+^n)\cong KO^*(S^0)[X]/(X^{n+1})$ and a ring isomorphism $H^*(HP_+^n;Z)\cong Z[x]/(x^{n+1})$, where $X_+=X\cup\{*\}$ denotes the disjoint union of the space X and the base point. We denote by $U\in KO^4(\Sigma(HP^{n-1})^\zeta)=KO^4(\Sigma Q_n)$ and $U_H\in H^4(\Sigma Q_n)$ the Thom classes of ζ in the respective cohomology theories, where U is the one defined by the method of [3]. Then we have the associated Thom isomorphism $E^*(HP_+^{n-1})\to E^{*+4}(\Sigma Q_n)$ for $E=KO$ and HZ, and thus $KO^*(\Sigma Q_n)\cong KO^*(S^0)\{UX^i\mid 0\leqq i\leqq n-1\}$ and $H^*(\Sigma Q_n;Z)\cong Z\{U_Hx^i\mid 0\leqq i\leqq n-1\}$. For $i\geqq 1$ we denote by $X(i)\in KO^{4i}(KP^n)$ the element X^i for the case $KP^n=HP^n$ and UX^{i-1} for the case $KP^n=\Sigma Q_n$ respectively. Since the collapsing map $p: KP^n\to KP_m^n$ induces a monomorphism $p^*: KO^*(KP_m^n)\to KO^*(KP^n)$, we have the following identification through p^*.

Lemma 1.1. $KO^*(KP_m^n)\cong KO^*(S^0)\{X(i)\mid m\leqq i\leqq n\}$.

For a $(c-1)$-connected covering spectrum F of a spectrum E, we have a canonical isomorphism $F^q(X)\cong \mathrm{Im}(E^q(X/X^{c+q})\to E^q(X/X^{c+q-1}))$ for any CW-spectrum X, where X^i denotes the i dimensional skeleton of X. Since kO and $kSpin$ are the -1 and 3 connected covering spectra of KO respectively, we have the following canonical isomorphisms:

Lemma 1.2. *Let* $0\leqq\varepsilon\leqq 3$.

(i) $kO^{4k-\varepsilon}(KP_m^n)\cong KO^{4k-\varepsilon}(KP_k^n)$ *if* $m\leqq k\leqq n$, *and* $kO^{4k-\varepsilon}(KP_m^n)\cong KO^{4k-\varepsilon}(KP_m^n)$ *if* $k\leqq m$.

(ii) $kSpin^{4k-\varepsilon}(KP_m^n)\cong KO^{4k-\varepsilon}(KP_{k+1}^n)$ *if* $m-1\leqq k\leqq n-1$, *and*

$kSpin^{4k-\varepsilon}(KP^n_m) \cong KO^{4k-\varepsilon}(KP^n_m)$ *if* $k \leqq m-1$.

Let $ph : KO^i(W_+) \to H^*(W_+ ; Q)$ be the Pontrjagin character, and $\sinh(y) = \frac{e^y - e^{-y}}{2} = \sum_{i\geqq 0} \frac{y^{2i+1}}{(2i+1)!} \in Q[y]$. Then the formula for $ph(X) \in H^*(HP^n_+ ; Q)$ is well known, and the formula for $ph(U) \in H^*(Q_n ; Q)$ can be deduced from the formula in [1] (see [7]).

Lemma 1.3.

(i) $ph(X) = \left(2\sinh\frac{\sqrt{x}}{2}\right)^2$ *in* $H^*(HP^n_+ ; Q) = Q[x]/(x^{n+1})$.

(ii) $ph(U) = U_H \frac{d}{dx}\left(2\sinh\frac{\sqrt{x}}{2}\right)^2$ *in* $H^*(Q_n ; Q) = Q[x]/(x^n)\{U_H\}$.

By the definition of A, we have an exact sequence

(1.4) $kO^{k-1}(KP^n_m) \xrightarrow{\psi} kSpin^{k-1}(KP^n_m) \to A^k(KP^n_m) \xrightarrow{d} kO^k(KP^n_m) \xrightarrow{\psi}$ $kSpin^k(KP^n_m)$,

where kO and $kSpin$ are assumed to be localized at 2. By Lemmas 1.1 and 1.2, the operation ψ in (1.4) is identified with the operation ψ^3-1 on the appropriate subgroup of $KO^i(KP^n)$.

2. Free parts. Let $\psi_j(Q) = \psi \otimes Q : kO^{4j}(KP^n_m) \otimes Q \to kSpin^{4j}(KP^n_m) \otimes Q$, where ψ is the homomorphism in (1.4). Then we have

Lemma 2.1. *$A^i(KP^n_m) \otimes Q \cong \mathrm{Ker}(\psi_j(Q)) \cong Q$ if $i = 4j$ and $m \leqq j \leqq n$, and $A^i(KP^n_m) \otimes Q \cong 0$ otherwise.*

Proof. By (1.4) and Lemmas 1.1 and 1.2, we have $A^{4j}(KP^n_m) \otimes Q \cong \mathrm{Ker}(\psi_j(Q))$, $A^{4j+1}(KP^n_m) \otimes Q \cong \mathrm{Coker}(\psi_j(Q))$ and $A^{4j+2}(KP^n_m) \otimes Q = A^{4j+3}(KP^n_m) \otimes Q = 0$. We show that $\mathrm{Ker}(\psi_j(Q)) \cong H^{4j}(KP^n_m ; Q)$. Then we have the desired result, because the value of $\mathrm{rank}(kO^{4j}(KP^n_m)) - \mathrm{rank}(kSpin^{4j}(KP^n_m))$ is equal to 1 if $m \leqq j \leqq n$, and equal to 0 otherwise. Since $\mathrm{Ker}(\psi_j(Q)) = 0$ if $j \geqq n+1$, we assume that $j \leqq n$. We have isomorphisms $kO^{4j}(KP^n_m) \otimes Q \cong KO^{4j}(KP^n_t) \otimes Q \cong \bigoplus_{i=j}^{n} H^{4i}(KP^n_t ; Q)$. Here the first isomorphism is induced by that of Lemma 1.2 and the second one is induced by the Pontrjagin character, and $t = j$ if $m \leqq j \leqq n$ and $t = m$ if $j \leqq m$. Similarly we have isomorphisms $kSpin^{4j}(KP^n_m) \otimes Q \cong KO^{4j}(KP^n_s)$

$\otimes Q \cong \bigoplus_{i=j}^{n} H^{4i}(KP_s^n; Q)$, where $s = j+1$ if $m-1 \leqq j \leqq n-1$ and $s = m$ if $j \leqq m-1$. Through these isomorphisms, $\psi_j(Q)$ is identified with a homomorphism $\psi_H : \bigoplus_{i=j}^{n} H^{4i}(KP_t^n; Q) \to \bigoplus_{i=j}^{n} H^{4i}(KP_s^n; Q)$ which maps a homogeneous element $x \in H^{4i}(KP_t^n; Q)$ to the element $(3^{4(i-j)}-1)x \in H^{4i}(KP_s^n; Q)$ if $i \geqq s$ and to 0 otherwise. Thus we have $\mathrm{Ker}(\psi_j(Q)) \cong H^{4j}(KP_m^n; Q)$, and we complete the proof. Q. E. D.

For an orientable vector bundle E over a connected finite CW-complex B with fiber dimension d, the A-theory codegree $cd^A(E; B) \in Z$ is the order of the $\mathrm{Coker}(i^* : A^d(E) \to A^d(S^d))$, where $i : S^d \to E$ is the inclusion of the fiber over the base point of B (cf. [5]). Using the A-codegree, we can describe a generator of the free part of $A^{4m}(KP_m^n)$ as follows :

Proposition 2.2. *We have a generator Y of the free part of $A^{4m}(KP_m^n)$, which satisfies the following, where d denotes the map $A^{4m}(KP_m^n) \to kO^{4m}(KP_m^n)_{(2)} \subset KO^{4m}(KP^n)_{(2)}$:*

(i) *For $KP_m^n = HP^n/HP^{m-1}$,*

$$d(Y) = cd^A(m\xi; HP^{n-m}) \sum_{j=0}^{n-m} \frac{r_j(m)}{a(j)} g_j X^{m+j},$$

where $r_j(m) \in Q$ is the coefficient in the expansion

$$\left(2\sinh^{-1}\frac{\sqrt{x}}{2}\right)^{2m} = \sum_{j \geqq 0} r_j(m) x^j.$$

(ii) *For $KP_m^n = \Sigma Q_n/Q_{m-1}$,*

$$d(Y) = cd^A(\zeta \oplus (m-1)\xi; HP^{n-m}) \sum_{j=0}^{n-m} \frac{s_j(m)}{a(j)} g_j UX^{m+j-1},$$

where $s_j(m) \in Q$ is the coefficient in the expansion

$$\left(\frac{1}{m}\right)\cdot\frac{d}{dy}\left(2\sinh^{-1}\frac{\sqrt{x}}{2}\right)^{2m} = \sum_{j \geqq 0} s_j(m) x^{m+j-1}.$$

Proof. This is simply the quaternionic version of the result in [5]. For a vector bundle E over a connected finite CW-complex B, which has a Thom class U_{KO}^E in the KO-cohomology and a Thom class U_H^E in the ordinary cohomology, $sh(E) \in KO^0(B_+) \otimes Q$ denotes the characteristic class defined by $ph(U_{KO}^E) = U_H^E sh(E)$, that is the KO-Todd class (cf. [1]). Then $ph(U_{KO}^E ph^{-1} sh(-E)) = U_H^E$, and we have $(\psi^3-1)(U_{KO}^E ph^{-1}(sh(-E)) = 0$, where $ph : KO^0(B_+) \otimes Q \to H^{4*}(B_+ ; Q)$ is an isomorphism. Consider the case $E = m\xi$ or $E = \zeta \oplus (m-1)\xi$, and $B = HP^{n-m}$. Then the Thom space of E is homeomorphic to KP_m^n ([2]), and $U_{KO}^E ph^{-1}(sh(-E))$ is a generator of $\mathrm{Ker}(\psi^3-1) \otimes Q$. Then we can take a generator Y of the free part of $A^{4m}(KP_m^n)$, which satisfies $d(Y) = c^E U_{KO}^E ph^{-1}(sh(-E))$. Here c^E is the minimal positive integer satisfying $d(Y) \in KO^{4m}(KP_m^n)_{(2)}$. But it is easy to see that $c^E = cd^A(E ; HP^{n-m})$ for $E = m\xi$ or $\zeta \oplus (m-1)\xi$. By Lemma 1.3, we have $sh(m\xi) = \left(2\sinh\frac{\sqrt{x}}{2}\right)^{2m}$ and $sh(\zeta \oplus (m-1)\xi) = \frac{1}{mx^{m-1}}\cdot\frac{d}{dx}\left(2\sinh\frac{\sqrt{x}}{2}\right)^{2m}$. Hence $ph^{-1}(sh(-m\xi)) = \sum_{j\geq 0}\frac{r_j(m)}{a(j)}g_j X^j$ and $ph^{-1}(sh(-(\zeta \oplus (m-1)\xi))) = \sum_{j\geq 0}\frac{s_j(m)}{a(j)}g_j X^j$. U_{KO}^E is identified with X^m or UX^{m-1} for $E = m\xi$ or $\zeta \oplus (m-1)\xi$ respectively, through the monomorphism $p^* : KO^{4m}(KP_{m-1}^n) \to KO^{4m}(KP^n)$. Thus we have the desired result.

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Since $A^{4i}(KP_m^n) = A^{4i}(KP_i^n)$ for $m \leq i \leq n$, Proposition 2.2 gives a description of the generators of the free parts for all cases.

3. Stable Adams operations. The formula for $\psi^k X$ is known as follows :

Lemma 3.1. (S. Feder-S. Gitler [6]). *Let* $X = [\xi-1] \in KO^4(HP^n)$. *Then*

$$\psi^k(X) = \sum_{i=1}^{k}\frac{1}{k\cdot i\cdot a(i+1)}\binom{k+i-1}{2i-1}g_{i-1}X^i \quad in\ KO^4(HP^n)\left[\frac{1}{k}\right].$$

Especially, we have $\psi^3(X) = X+(g_1/3)X^2+(g_2/9)X^3$ *in* $KO^4(HP^n)_{(2)}$.

We need the following

Lemma 3.2. *Let U be the KO-Thom class of ζ over HP^{n-1}. Then*

$$\psi^k(U) = \sum_{i=1}^{k}\frac{1}{k}\binom{k+i-1}{2i-1}g_{i-1}UX^{i-1} \quad in\ KO^3(Q_n)\left[\frac{1}{k}\right].$$

Especially, we have $\psi^3(U) = U+(2g_1/3)UX+(g_2/3)UX^2$ *in* $KO^3(Q_n)_{(2)}$.

Proof. Since $ph : KO^3(Q_n)\left[\frac{1}{k}\right] \to H^*(Q_n ; Q)$ is a monomorphism, we may show the equality $ph(\psi^k(U)) = \sum_{i=1}^{k} \frac{1}{k}\binom{k+i-1}{2i-1} ph(UX^{i-1})$. Using Lemma 1.3, we have $ph(\psi^k U) = U_H \frac{1}{k^2}\cdot\frac{d}{dx}\left(2\sinh\frac{k\sqrt{x}}{2}\right)^2$, and $ph(UX^{i-1})$ can be written as a polynomial of x over Q. Then we can compare the both sides of the desired equation, and assure the equality by easy arithmetic.

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Corollary 3.3. *Let* $\varepsilon = 1$ *or* 2, *and* $\psi^3 : KO^{4k-\varepsilon}(KP^n) \to KO^{4k-\varepsilon}(KP^n)$. *Then we have*

$$\psi^3(\alpha^\varepsilon g_{2j} X(k+2j)) = \sum_{i\geq 0} \binom{m+2j}{i} \alpha^\varepsilon g_{2(i+j)} X(k+2(i+j)),$$

where $X(t) = X^t$ *if* $KP^n = HP^n$ *and* $X(t) = UX^{t-1}$ *if* $KP^n = \Sigma Q_n$.

Corollary 3.4. *Let* $\psi^3 : KO^{4k}(KP^n) \otimes Z/2 \to KO^{4k}(KP^n) \otimes Z/2$. *Then*

(i) $$\psi^3(g_{2j} X^{k+2j}) = \sum_{i\geq 0} \binom{k+2j}{i} g_{2(i+j)} X^{k+2(i+j)} + k\sum_{i\geq 1} \binom{k+2j-1}{i-1} g_{2(i+j)-1} X^{k+2(i+j)-1},$$

$$\psi^3(g_{2j} UX^{k+2j-1}) = \sum_{i\geq 0} \binom{k+2j}{i} g_{2(i+j)} UX^{k+2(i+j)-1} + (k-1)\sum_{i\geq 1} \binom{k+2j-1}{i-1} g_{2(i+j)-1} UX^{k+2(i+j)-2}.$$

(ii) $$\psi^3(g_{2j+1} X(k+2j+1)) = \sum_{i\geq 0} \binom{k+2j+1}{i} g_{2(i+j)+1} X(k+2(i+j)+1).$$

We prepare the following notations, in which $[t]$ (resp. $\langle t\rangle$) denotes the maximal (resp. minimal) integer less than (resp. greater than) or equal to a rational number t :

$L(i, j) = \left[\frac{i-j}{2}\right]$ if $i \geqq j$, and $L(i, j) = 0$ otherwise.

$L'(i, j) = L(i, j+1)$.

$l_0(i, j) = \left\langle \frac{i-j}{2} \right\rangle$ if $i \geqq j$, and $l_0(i, j) = 0$ otherwise.
$l_0'(i, j) = l_0(i, j+1)$.
$m_0(i, j) = l_0(i, j)$ if $i \geqq j+2$, and $m_0(i, j) = 1$ otherwise.

Using these notations, the groups $kO^j(KP_m^n)$ and $kSpin^j(KP_m^n)$ are represented as subgroups of $KO^j(KP^n)$ as follows :

Lemma 3.5. *Let $L = L(n, k)$, $L' = L(n, k)$, $l_0 = l_0(m, k)$, $l_0' = l_0'(m, k)$ and $m_0 = m_0(m, k)$. Then we have*

(i) $kO^{4k}(KP_m^n) \cong \bigoplus_{j=l_0}^{L} Z\{g_{2j}X(k+2j)\} \oplus \bigoplus_{j=l_0'}^{L'} Z\{g_{2j+1}X(k+2j+1)\}$,

$kO^{4k-\varepsilon}(KP_m^n) \cong \bigoplus_{j=l_0}^{L} Z/2\{\alpha^\varepsilon g_{2j}X(k+2j)\}$ *for* $\varepsilon = 1$ *and* 2, *and*
$kO^{4k-3}(KP_m^n) = 0$.

(ii) $kSpin^{4k}(KP_m^n) \cong \bigoplus_{j=m_0}^{L} Z\{g_{2j}X(k+2j)\} \oplus \bigoplus_{j=l_0'}^{L'} Z\{g_{2j+1}X(k+2j+1)\}$,

$kSpin^{4k-\varepsilon}(KP_m^n) \cong \bigoplus_{j=m_0}^{L} Z/2\{\alpha^\varepsilon g_{2j}X(k+2j)\}$ *for* $\varepsilon = 1$ *and* 2,
and
$kSpin^{4k-3}(KP_m^n) = 0$.

Corollary 3.6. *For $\varepsilon = 1$ or 2 and $m \leqq k \leqq n-2$, let $M = (m_{i,j})$ be the matrix of $\psi^3-1 : kO^{4k-\varepsilon}(KP_m^n) \to kSpin^{4k-\varepsilon}(KP_m^n)$ with respect to the bases given in Lemma 3.5. Then M consists of the $L(n, k)$ rows and the $L(n, k)+1$ columns, and $m_{t,s} = \binom{k+2s-2}{t-s+1}$ if $t \geqq s$ and $m_{t,s} = 0$ if $t < s$.*

Corollary 3.7. *For $m \leqq k \leqq n-2$, let $M = (m_{i,j})$ be the matrix of $\psi^3-1 : kO^{4k}(KP_m^n) \otimes Z/2 \to kSpin^{4k}(KP_m^n) \otimes Z/2$ with respect to the bases which are the* mod 2 *reduction of those given in Lemma 3.5. Then M consists of the $(L(n, k)+L'(n, k)+1)$ rows and the $(L(n, k)+L'(n, k)+2)$ columns, and (t, s)-element $m_{t,s}$ is given by*

(1) $m_{t,s} = \binom{k+2s-2}{t-s+1}$ *if* $s \leqq t \leqq L$,

(2) $m_{t,s} = 0$ *if* $t < s$, $s = L+1$ *or* $s = L+L'+2$,

(3) $m_{L+1,s} = 0$ *if* $s \geqq 2$, *and* $m_{L+1,1} = k$ (*resp.* $k-1$) *for* $KP_m^n = HP_m^n$ (*resp.* $\Sigma Q_n/Q_{m-1}$),

(4) $m_{t,s} = \binom{k+2(s-L)-3}{t-s+1}$ *if* $L+2 \leqq t \leqq L+L'+1$ *and* $L+2 \leqq s \leqq L+L'+1$,
where $L = L(n, k)$ *and* $L' = L'(n, k)$.

By analyzing the matrices M in Corollaries 3.6 and 3.7, we can determine the bases of the kernel and the cokernel for the homomorphism ψ: $kO^i(KP_m^n) \otimes Z/2 \to kSpin^i(KP_m^n) \otimes Z/2$. Thus we can describe $A^i(KP_m^n)$ for $i \not\equiv 1 \bmod 4$ and $A^{4i+1}(KP_m^n) \otimes Z/2$ by (1.4). We state the result in the next section, where we restrict to $A^i(KP_m^n)$ for the case $4m-2 \leqq i \leqq 4n+1$, the calculation is essentially same for the case $i \leqq 4m-3$, and $A^i(KP_m^n) = 0$ for $i \geqq 4n+2$.

4. Results. For an integer i, $\nu(i)$ denotes the exponent of 2 in the prime power decomposition of i. In this section, we assume that the integers m, k and n satisfy $m \leqq k \leqq n$, and we put $L = \left[\frac{n-k}{2}\right]$. We prepare the following notations :

(1) The series $\{a_j\}$; a_j are the integers defined recursively by $a_1 = \nu(k)$ and $a_j = \nu\left(k+\sum_{t=1}^{j-1} 2^{a_t}\right)$.

(2) The integer s ; which is the minimal positive integer satisfying $2^{a_s}+\sum_{t=1}^{s-1} 2^{a_t-1} > L$.

(3) The integer K ; which is the maximal integer satisfying $2^K+\sum_{t=1}^{s-1} 2^{a_t} \leqq 2L$ if $s \geqq 2$ and $2^{K-1} \leqq L$ if $s = 1$.

(4) The set I ; $I = \{2^i \mid 0 \leqq i \leqq a_1-1\} \cup \bigcup_{t=1}^{s-2} \left\{2^i+\sum_{j=1}^{t} 2^{a_j-1} \,\middle|\, a_t \leqq i \leqq a_{t+1}-1\right\} \cup \left\{2^i+\sum_{j=1}^{s-1} 2^{a_j-1} \,\middle|\, a_{s-1} \leqq i \leqq K-1\right\}$ if $s \geqq 2$, and $I = \{2^i \mid 0 \leqq i \leqq K-1\}$ if $s = 1$.

(5) The set J ; $J = \{B\} \cup \{i \mid B < i \leqq L$ and $i+2^{\nu(i-B)+1} > L\}$, where $B = \sum_{t=1}^{s-1} 2^{a_t-1}$ if $s \geqq 2$ and $B = 0$ if $s = 1$.

Remark. By the definitions (1)—(3) above, we have $a_{s-1} \leqq K \leqq$

$[\log_2 L]+1$, where $a_0 = 1$ if $s = 1$.

(I) **$A^{4k}(KP^n_m)$.** By (1.4), we have $A^{4k}(KP^n_m) \cong Z_{(2)}\{Y\} \oplus \mathrm{Coker}(\psi)$, where Y is the element given in Proposition 2.2 and $\mathrm{Coker}(\psi)$ denotes the cokernel of $\psi: kO^{4k-1}(KP^n_m) \to kSpin^{4k-1}(KP^n_m)$.

Theorem 4.1. *Assume that $m \leqq k \leqq n$.*

(i) *If k is odd or $k \geqq n-1$, then $A^{4k}(KP^n_m) \cong Z_{(2)}$.*

(ii) *If k is even and $k \leqq n-2$, then $A^{4k}(KP^n_m) \cong Z_{(2)} \oplus (Z/2)^K$, and we can take $\{\alpha g_{2i}X(k+2i) \mid i \in I\}$ as a basis of $(Z/2)^K \cong \mathrm{Coker}(\psi)$, where the integer K and the set I are those in (3) and (4) respectively defined for k, n and m.*

(II) **$A^{4k-\varepsilon}(KP^n_m)$ for $\varepsilon = 1$ or 2.** By (1.4), we have a short exact sequence

$$0 \to \mathrm{Coker}(\psi_{4k-\varepsilon-1}) \to A^{4k-\varepsilon}(KP^n_m) \xrightarrow{d} \mathrm{Ker}(\psi_{4k-\varepsilon}) \to 0,$$

where ψ_i denotes $\psi: kO^i(KP^n_m) \to kSpin^i(KP^n_m)$.

Theorem 4.2. *Let $\varepsilon = 1$ or 2, and let K, I and J be as in (3), (4) and (5) respectively defined for k, n and m. Then we have*

(i) *If k is odd or $L = 0$, we have an isomorphism*

$$d: A^{4k-\varepsilon}(KP^n_m) \to \mathrm{Ker}(\psi_{4k-\varepsilon}) = Z/2\{\alpha^\varepsilon g_{2L}X(k+2L)\}.$$

(ii) *If k is even, $L \geqq 1$ and $\varepsilon = 1$, we have $\mathrm{Coker}(\psi_{4k-2}) \cong (Z/2)^K$ whose basis is given by $\{\alpha^2 g_{2i}X(k+2i) \mid i \in I\}$, and $\mathrm{Ker}(\psi_{4k-1}) \cong (Z/2)^{K+1}$ whose basis is given by $\{\alpha g_{2i}X(k+2i) \mid i \in J\}$.*

(iii) *If k is even, $L \geqq 1$ and $\varepsilon = 2$, we have an isomorphism*

$$d: A^{4k-2}(KP^n_m) \to \mathrm{Ker}(\psi_{4k-2}) \cong (Z/2)^{K+1},$$

and we can take $\{\alpha^2 g_{2i}X(k+2i) \mid i \in J\}$ as a basis.

(III) **$A^{4k+1}(KP^n_m) \otimes Z/2$.** By (1.4), we have $A^{4k+1}(KP^n_m) \cong \mathrm{Coker}(\psi: kO^{4k}(KP^n_m)_{(2)} \to kSpin^{4k}(KP^n_m)_{(2)})$. For the case $k = n$ or $n-1$, we have $A^{4n+1}(KP^n_m) = 0$, and $A^{4n-3}(KP^n_m) \cong Z/(n-1)$ by an easy calculation. But, for the case $k \leqq n-2$, it is not easy to determine explicit generators and their orders of the cyclic summands of $A^{4k+1}(KP^n_m)$. We restrict the problem to calculate the group $A^{4k+1}(KP^n_m) \otimes Z/2$ for $m \leqq k \leqq n-2$, which only enumerates the number of the direct summands. Then we have the result by considering the matrix in Corollary 3.7. For the statement of the result, we need the notations $\bar{K}$ and $\bar{I}$ which are equal to K and I respectively defined for $k+1$, n and m.

Theorem 4.3. *Let $m \leqq k \leqq n-2$. Then we have*

(i) *If k is even, then $A^{4k+1}(KP_m^n) \otimes Z/2 \cong (Z/2)^{K+1}$ whose basis is given by $\{g_{2j}X(k+2j) \mid j \in I\} \cup \{g_1 X(k+1)\}$.*

(ii) *If k is odd, then $A^{4k+1}(KP_m^n) \otimes Z/2 \cong (Z/2)^{\bar{K}+1}$ whose basis is given by $\{g_{2j+1}X(k+2j+1) \mid j \in \{0\} \cup \bar{I}\}$.*

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