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ON H-RINGS WITH HOMOGENEOUS SOCLES

Dedicated to Professor Takasi Nagahara on his 60th birthday

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In [5], we introduced Harada (H-) rings and co-Harada (co-H-) rings. However, in later [7], we showed that left H-rings are the same with right co-H-rings. These rings include QF-rings and generalized uniserial rings ([5], [6]). The purpose of the present paper is to study left H-rings with homogeneous socles. In Theorems 1 and 2, we show that such a ring is constructed by a suitable local QF-ring and has the self-duality in the sense of Morita.

1. Preliminaries. Throughout this paper, we assume that R is an associative ring with identity, all R-modules are unitary and all homomorphisms between R-modules are written on the opposite side of scalars. The notation M_R (resp. ${}_RM$) is used to stress that M is a right (resp. left) R-module. For an R-module M, we use $\mathrm{E}(M)$, $\mathrm{J}(M)$, $\mathrm{Soc}(M)$ and $Z(M)$ to denote its injective hull, Jacobson radical, socle and singular submodule, respectively. Further, by $\{\mathrm{J}_i(M)\}$ and $\{\mathrm{S}_i(M)\}$, we denote the descending Loewy chain and the ascending Loewy chain of M. respectively. For two R-modules M and N, we use '$M \subseteq N$' to stand for that M is isomorphic to a submodule of N. An R-module M is said to be a small module if it is a small submodule of its injective hull, and M is said to be a non-small module if it is not a small module ([3]).

Now, a ring R is said to be a left H-ring if it is a left artinian ring and, for any primitive idempotent e in R such that ${}_RRe$ is non-small, there exists $t \geqq 0$ such that

1) ${}_RRe/\mathrm{S}_k({}_RRe)$ is injective for $0 \leqq k \leqq t$, and
2) ${}_RRe/\mathrm{S}_{t+1}({}_RRe)$ is a small module.

Dually, a ring R is said to be a right co-H-ring if it satisfies the ascending chain condition on right annihilator ideals and there is a partition $\{e_i\} \cup \{f_j\}$ of a complete set of orthogonal primitive idempotents of R such that

1) each e_iR_R is injective and each f_jR_R is a small module,
2) for each e_iR, there exists t_i such that $\mathrm{J}_t(e_iR_R)_R$ is projective for $0 \leqq t \leqq t_i$, and $\mathrm{J}_{t_i+1}(e_iR_R)_R$ is a singular module, and

3) for each f_j, there exists e_iR for which $f_jR_R \subsetneqq e_iR_R$.
However, in [7], we showed that left H-rings and right co-H-rings are the same rings. We further see from [7, Proposition 3.6] that, for a left H-ring R with a complete set E of primitive idempotents, there exists a one to one onto mapping between $\{e \in E \mid eR_R \text{ is injective}\}$ and $\{f \in E \mid {}_RRf \text{ is injective}\}$. So we see that, for a left H-ring, its right socle is homogeneous if and only if its left socle is homogeneous.

A ring R is said to be self-dual (or have the self-duality) if there exists a Morita duality between the category of finitely generated left R-modules and the category of finitely generated right R-modules. For a left artinian ring R with a finitely generated injective co-generator ${}_RE$, it is well known ([1], [4]) that the functor $\mathrm{Hom}_R(\ \ , E)$ and $\mathrm{Hom}_T(\ \ , E)$ where $T = \mathrm{End}_R(E)$, give a duality between the category of finitely generated left R-modules and the one of finitely generated right T-modules. It is also known that a basic left artinian ring R has the self-duality if and only if its minimal injective co-generator ${}_RE$ is finitely generated and $\mathrm{End}_R(E)$ is isomorphic to R.

2. Let Q be a local QF-ring, and put $J = \mathrm{J}(Q)$ and $S = \mathrm{Soc}(Q_Q)$ $(= \mathrm{Soc}({}_QQ))$. We define $W = W_n^k(Q)$ by setting

$$W_n^k(Q) = \begin{pmatrix} \overbrace{Q & \cdots & Q}^{k} & \bar{Q} & \cdots & \bar{Q} \\ J & \ddots & \vdots & \vdots & & \vdots \\ \vdots & \ddots & Q & \vdots & & \vdots \\ \vdots & & J & \bar{Q} & & \vdots \\ \vdots & & \vdots & \bar{J} & \ddots & \vdots \\ \vdots & & \vdots & \vdots & \ddots\ \bar{Q} & \vdots \\ J & \cdots & J & \bar{J} & \cdots\ \bar{J} & \bar{Q} \end{pmatrix}$$

where $\bar{Q} = Q/S$ and $\bar{J} = J/S$. Since $JS = SJ = 0$, $W_n^k(Q)$ canonically becomes a ring. We put

$$e_i = \begin{pmatrix} 0 & & & & \\ & \ddots & & 0 & \\ & & 0 & & \\ & & & 1_{(i)} & \\ & 0 & & 0 & \\ & & & & \ddots \\ & & & & & 0 \end{pmatrix} \quad \text{in } W = W_n^k(Q)$$

for $i = 1,\dots, n$, and put

$$X_j = \begin{pmatrix} & \overbrace{\begin{matrix} 0 \\ \vdots \\ 0 \end{matrix}}^{k} \Big\} j & \\ 0 & S & 0 \\ & 0 & \\ & \vdots & \\ & 0 & \end{pmatrix}$$

for $J = 1,\dots, n$. Then, $\{e_1,\dots, e_n\}$ is a complete set of orthogonal primitive idempotents and the following are easily verified :

1) ${}_{e_iWe_i}e_iWe_j$ and $e_iWe_{j\,e_jWe_j}$ are artinian for all e_i and e_j ; so W is left and right artinian.

2) $\mathrm{Soc}(e_iW_W) = X_i$ for $i = 1,\dots, n$.

3) $(e_1W_W, {}_WWe_k)$ is an injective pair ; so e_1W_W and ${}_WWe_k$ are injective (cf. [2], [7, Lemma 1.1]).

4) $\mathrm{J}_i(e_1W_W)_W \simeq e_{i+1}W_W$ for $i = 1,\dots, n-1$.

5) $\mathrm{J}(e_nW_W)\mathrm{Soc}(W_W) = 0$; whence $\mathrm{J}(e_nW_W)_W$ is singular.

Therefore W is a right co-H-ring with the homogeneous right socle and hence W is a left H-ring with the homogeneous left socle ; in fact, the following conditions hold :

6) $\mathrm{S}_i({}_WWe_k) = X_1+\dots+X_i$ for $i = 1,\dots, n$.

7) ${}_WWe_k/\mathrm{S}_i({}_WWe_k)$ is injective for $i = 1,\dots, n-1$.

Now, our purpose of this section is to show the following result.

Theorem 1. *Let R be a basic left H-ring with the homogeneous left socle. Then R is represented as $R \simeq W_n^k(Q)$ for a suitable local QF-ring Q.*

Proof. As R is a basic right co-H-ring with the homogeneous socle, we can take a complete set $\{e_1,\dots, e_n\}$ of orthogonal primitive idempotents of R such that

1) e_1R_R is injective,

2) there is an isomorphism $\theta_{i,i-1}$ from e_iR_R to $\mathrm{J}_{i-1}(e_1R_R)_R$ for $i = 2,\dots, n$.

Let us take e_k in $\{e_1,\dots, e_n\}$ such that e_kR_R is a projective cover of $\mathrm{Soc}(e_1R_R)$ ($\simeq \mathrm{Soc}(e_iR)_R$ for all i). We claim that e_1Re_1 is a local QF-ring and R is isomorphic to $W_n^k(e_1Re_1)$.

We observe R by representing it as

$$R=\begin{pmatrix}[e_1, e_1] \cdots [e_n, e_1]\\ \vdots \qquad \vdots\\ [e_1, e_n] \cdots [e_n, e_n]\end{pmatrix}$$
$$=\begin{pmatrix}e_1Re_1 \cdots e_1Re_n\\ \vdots \qquad \vdots\\ e_nRe_1 \cdots e_nRe_n\end{pmatrix}$$

where $[e_i, e_j]$ denotes $\mathrm{Hom}_R(e_iR, e_jR)$.
We define a mapping $\Phi_{i,i+1}: [e_i, e_i] \to [e_{i+1}, e_{i+1}]$ by the rule: $\Phi_{i,i+1}(\alpha) = (\theta_{i+1,i})^{-1}\alpha\theta_{i+1,i}$ for $\alpha \in [e_i, e_i]$, and put $\Phi_{i,j} = \Phi_{j-1,j}\cdots\Phi_{i+1,i+2}\Phi_{i,i+1}$ for $i<j$. As is easily seen, $\Phi_{i,i+1}$ is a ring epimorphism and, by [7, Proposition 3.1], $\Phi_{i,i+1}$ is an isomorphism for $i \neq k$ but $\Phi_{k,k+1}$ is a non-isomorphism since

$$\mathrm{Ker}\,\Phi_{k,k+1} = \{\alpha \in [e_k, e_k] \mid \mathrm{Im}\,\alpha \subseteq \mathrm{Soc}(e_kR_R)\}.$$

Put $X = \mathrm{Ker}\,\Phi_{1,k+1}$. Then ${}_{e_1Re_1}X$ and $X_{e_1Re_1}$ are simple, and moreover $X = \mathrm{Soc}({}_{e_1Re_1}e_1Re_1) = \mathrm{Soc}(e_1Re_{1e_1Re_1})$ (cf. [7, Lemma 3.3]). As a result, we see that e_1Re_1 is a local QF-ring, and

$$e_1Re_1 \overset{\Phi_{1,i}}{\simeq} e_iRe_i \text{ for } i \leqq k$$

and

$$e_1Re_1/X \overset{\overline{\Phi}_{1,i}}{\simeq} e_iRe_i \text{ for } i > k$$

where $\Phi_{1,i}$ is an indued isomorphism by $\Phi_{1,i}$.

For $j \geqq i$, we define a mapping $\xi_{j,i}: [e_j, e_j] \to [e_j, e_i]$ given by $\xi_{j,i}(\alpha) = \theta_{j,i}^{-1}\alpha$ for $\alpha \in [e_j, e_j]$. Clearly, $\xi_{j,i}$ is an isomorphism as an abelian group.

For $j<i$, we define $\eta_{j,i}: \mathrm{J}([e_j, e_j]) \to [e_j, e_i]$ by the rule $\eta_{j,i}(\alpha) = \theta_{i,j}^{-1}\alpha$ for $\alpha \in \mathrm{J}([e_j, e_j])$. Of course, this mapping is well dedined, because $\mathrm{Im}\,\alpha \subsetneqq \theta_{n,j}(e_nR) \subseteq \theta_{i,j}(e_iR)$ for $\alpha \in [e_j, e_j]$. Clearly $\eta_{j,i}$ is also an isomorphism as an abelian group.

Here we define $\{\phi_{ij} \mid i=1,\ldots,n,\ j=1,\ldots,n\}$ by putting

$$\phi_{ij}=\begin{cases}\xi_{j,i}\Phi_{1,j} & \text{for } k \geqq j,\ j \geqq i\\ \eta_{j,i}\Phi_{1,j} & \text{for } k \geqq j,\ i > j\\ \xi_{j,i}\overline{\Phi}_{1,j} & \text{for } j > k,\ j \geqq i\\ \eta_{j,i}\overline{\Phi}_{1,j} & \text{for } j > k,\ i > j\end{cases}$$

Then

$$e_1Re_1 \overset{\phi_{ij}}{\simeq} [e_j, e_i] \text{ for } k \geqq j,\ j \geqq i,$$

$$J(e_1Re_1) \overset{\phi_{ij}}{\simeq} [e_j, e_i] \text{ for } k \geqq j,\ i > j,$$

$$e_1Re_i/\mathrm{Soc}(e_1Re_1) \overset{\phi_{ij}}{\simeq} [e_j, e_i] \text{ for } k < j,\ j \geqq i, \text{ and}$$

$$J(e_1Re_1)/\mathrm{Soc}(e_1Re_1) \overset{\phi_{ij}}{\simeq} [e_j, e_i] \text{ for } k < j,\ i > j.$$

Now, it is straightforward to verify that $\phi_{ij}\phi_{jk} = \phi_{ik}$ for all i, j, k. Hence it follows that R is isomorphic to $W_n^k(e_1Re_1)$ by the mapping :

$$\begin{pmatrix} \alpha_{11} & \cdots & \alpha_{1n} \\ \vdots & & \vdots \\ \alpha_{n1} & \cdots & \alpha_{nn} \end{pmatrix} \to \begin{pmatrix} \phi_{11}(\alpha_{11}) & \cdots & \phi_{1n}(\alpha_{1n}) \\ \vdots & & \vdots \\ \phi_{n1}(\alpha_{n1}) & \cdots & \phi_{nn}(\alpha_{nn}) \end{pmatrix}$$

This completes the proof.

Corollary. *If R is a basic generalized uniserial ring with homogeneous socle, then $R \simeq W_n^k(Q)$ for a suitable local uniserial ring Q.*

3. As mentioned in the introduction, in this section, we show the following

Theorem 2. *Left H-rings with homogeneous left socle have self-duality.*

Proof. Let R be a left H-ring with homogeneous left socle. We can assume that R is basic. As R is a right co-H-ring with homogeneous right socle, we can take a complete set $E = \{e_1, \dots, e_n\}$ of orthogonal primitive idempotents such that

1) e_1R_R is injective,

2) $e_1R_R \supsetneqq e_2R_R \supsetneqq \cdots \supsetneqq e_nR_R$; more precisely, $J(e_iR_R)_R \simeq e_{i+1}R_R$ for $i = 1, \dots, n-1$.

We represent R as

$$R = \begin{pmatrix} [e_1, e_1] & \cdots & [e_n, e_1] \\ \vdots & & \vdots \\ [e_1, e_n] & \cdots & [e_n, e_n] \end{pmatrix}$$

where $[e_i, e_j] = \mathrm{Hom}_R(e_iR, e_jR)$.

We can take e_k in E such that e_kR_R is the projective cover of

$\mathrm{Soc}(e_1R_R)$ ($\simeq \mathrm{Soc}(e_iR_R)$ for all i). Put $X_i = \mathrm{Hom}_R(e_kR, \mathrm{Soc}(e_iR))$ for $i = 1,\dots, n$. Then, by [7], the following hold:

$$3)\quad \mathrm{Soc}(e_iR_R) = \begin{pmatrix} & 0 & \\ & \vdots & \\ & 0 & \\ 0 & X_i & 0 \\ & 0 & \\ & \vdots & \\ & 0 & \end{pmatrix}$$

for $i = 1,\dots, n$, and

$$4)\quad \mathrm{S}_i({}_RRe_k) = \begin{pmatrix} & X_1 & \\ & \vdots & \\ 0 & X_i & 0 \\ & 0 & \\ & \vdots & \\ & 0 & \end{pmatrix}$$

for $i = 1,\dots, n$ and

5) ${}_RRe_k/\mathrm{S}_i({}_RRe_k)$ is injective for $i = 1,\dots, n-1$.

We put $E_i = {}_RRe_k/\mathrm{S}_{i-1}({}_RRe_k)$ for $i = 1,\dots, n$, and put $I = E_1 \oplus \dots \oplus E_n$. Since $\{E_i\}$ is the representative set of indecomposable injective left R-modules, we see that I is a minimal injective co-generator. Therefore, we may show that R is isomorphic to $T = \mathrm{End}_R(I)$.

For a finitely generated left R-module M, we put $M^* = \mathrm{Hom}_R(M, I)$. Since R is a left H-ring, we see that T is a right co-H-ring. Here we represent T as

$$\mathrm{T} = \begin{pmatrix} [E_1, E_1] & \cdots & [E_1, E_n] \\ \vdots & & \vdots \\ [E_n, E_1] & \cdots & [E_n, E_n] \end{pmatrix}$$

where $[E_i, E_j] = \mathrm{Hom}_R(E_i, E_j)$ for all i, j. And, put

$$g_i = \begin{pmatrix} 0 & & & & & \\ & \ddots & & & 0 & \\ & & 0 & & & \\ & & & 1 & & \\ & 0 & & & 0 & \\ & & & & & \ddots \\ & & & & & & 0 \end{pmatrix} \quad (1 \text{ in the } (i,i) \text{ position})$$

for $i = 1, \dots, n$. Then, $\{g_i\}$ is a complete set of orthogonal primitive idempotents of T. Now, we can identify

$$g_i T = E_i^* = \begin{pmatrix} & 0 & \\ [E_i, E_1] & \cdots & [E_i, E_n] \\ & 0 & \end{pmatrix}$$

for $i = 1, \dots, n$. By 5), we see that

6) $g_1 T_T$ is injective,

7) $g_1 T_T \supsetneqq g_2 T_T \supsetneqq \cdots \supsetneqq g_n T_T$.

Here, in view of 3), 4), we see that

$$\mathrm{Soc}(g_1 T_T) = \begin{pmatrix} 0 \dots 0 & [E_1, \mathrm{Soc}(E_k)] & 0 \cdots 0 \\ & 0 & \\ 0 & \vdots & 0 \\ & \vdots & \\ & 0 & \end{pmatrix}$$

where $[E_1, \mathrm{Soc}(E_k)] = \mathrm{Hom}_T(E_1, \mathrm{Soc}(E_k))$. Hence $g_k T$ is a projective cover of $\mathrm{Soc}(g_1 T_T)$. Hence, by Theorem 1 and its proof, we see that $T \simeq W_n^k(g_1 T g_1)$. Since $g_1 T g_1 \simeq [E_1, E_1] \simeq e_1 R e_1$ and $R \simeq W_n^k(e_1 R e_1)$, it follows that T is isomorphic to R as required.

References

[1] G. Azumaya : A duality theory for injective modules, Amer. J. Math. 81 (1959), 249–278.

[2] K. R. Fuller: On indecomposable injectives over artinian rings, Pacific J. Math. 29 (1960), 115–135.

[3] M. Harada : Non-small modules and non-cosmall modules, In Ring Theory : Proceedings of the 1978 Antwerp conference, F. Van Oystaeyen, ed. New York : Marcel Dekker, 1979, 669–687.

[4] K. Morita : Duality for modules and its applications to the theory of rings with minimal condition, Sci. Rep. Tokyo Kyoiku Daigaku Sec. A 6 (1958), 83–142.

[5] K. Oshiro : Lifting modules, extending modules and their applications to QF-rings, Hokkaido Math. J. 13 (1984), 310–338.

[6] K. Oshiro : Lifting modules, extending modules and their applications to generalized uniserial rings, Hokkaido Math. J. 13 (1984), 339–346.

[7] K. Oshiro : On Harada rings. I, Math. J. Okayama Univ. 31 (1989), 161–178.

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