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ON HARADA RINGS. II

Dedicated to Professor Hiroyuki Tachikawa on his 60th birthday

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Let R be a left artinian ring and E a complete set of orthogonal primitive idempotents. R is said to be a left Harada (H-) ring if E is arranged as

$$E = \{e_{11}, \dots, e_{1n(1)}, e_{21}, \dots, e_{2n(2)}, \dots, e_{m1}, \dots, e_{mn(m)}\}$$

for which

1) each $e_{i1}R$ is an injective right R-module,

2) $e_{i,j-1}R \simeq e_{ij}R$ or $e_{i,j-1}J \simeq e_{ij}R$ for $1 \le i \le m$, $1 < j \le n(i)$ where J is the Jacobson radical of R.

In [4] (cf. [3]), we showed that, for a left artinian ring R, the following conditions are equivalent :

1) R is a left Harada ring.

2) The family of all injective left R-modules is closed under taking small covers.

3) The family of all projective right R-modules is closed under taking essential extensions.

In the present paper, we shall show that left H-rings can be represented as suitable generalized matrix rings. As left H-rings are Morita invariant, we may restrict our attention to basic left H-rings.

Notation. Throughout this paper, rings R considered are associative rings with identity and all R-modules are unitary. The notation M_R (resp. M_R) is used to stress that M is a right (resp. left) R-module. For an R-module M, $J(M)$ and $S(M)$ denote its Jacobson radical and socle, respectively, and $\{J_i(M)\}$ and $\{S_i(M)\}$ denote its descending Loewy chain and ascending Loewy chain, respectively.

For R-modules M and N, for the sake of convenience, we put

$$(M, N) = \mathrm{Hom}_R(M, N)$$

and in particular, put

$$(e, f) = (eR, fR) = \mathrm{Hom}_R(eR, fR)$$

Now, henceforth, we assume that R is a basic H-ring and E a complete set of orthogonal primitive idempotents of R. So, E is arranged as

$$E = \{e_{11}, \ldots, e_{1n(1)}, \ldots, e_{m1}, \ldots, e_{mn(m)}\}$$

for which

1) each $e_{i1}R_R$ is injective,

2) there exists an isomorphism $\theta^i_{k,k-1}$ from $e_{ik}R_R$ to $e_{i,k-1}J(R)_R = J(e_{i,k-1}R_R)$.

We put

$$\theta^i_{1,1} = \text{the identity map of } e_{i1}R_R$$
$$\theta^i_{k,1} = \theta^i_{2,1}\cdots\theta^i_{k-1,k-2}\theta^i_{k,k-1}$$

We represent R as

$$R = \begin{pmatrix} (e_{11}, e_{11}) & \cdots (e_{mn(m)}, e_{11}) \\ & \cdots \\ (e_{11}, e_{mn(m)}) & \cdots (e_{mn(m)}, e_{mn(m)}) \end{pmatrix}$$
$$= \begin{pmatrix} e_{11}Re_{11} & \cdots e_{11}Re_{mn(m)} \\ & \cdots \\ e_{mn(m)}Re_{11} & \cdots e_{mn(m)}Re_{mn(m)} \end{pmatrix}$$

The following properties hold on R ([4]) :

a) Each $S(e_{ij}R_R)_R$ is simple,

$$S(e_{i1}R_R)_R \simeq \cdots \simeq S(e_{in(i)}R)_R,$$
$$S(e_{ij}R_R)_R \not\simeq S(e_{kt}R)_R \text{ if } i \neq k,$$

b) For each e_{i1}, there exists unique $g_i \in E$ such that $(e_{i1}R_R\,;\,{}_RRg_i)$ is an injective pair, i.e., $g_iR_R/J(g_iR_R)_R \simeq S(e_{i1}R_R)_R$ and ${}_RRe_{i1}/J({}_RRe_{i1}) \simeq {}_RS({}_RRg_i)$. Then ${}_RRg_i$ is injective, and

$$S_k({}_RRg_i) = S(e_{i1}R_R) + \cdots + S(e_{ik}R_R)$$

for $1 \leq i \leq m$, $1 \leq k \leq n(i)$. So, $S_k(Rg_i)$ is a two-sided ideal. In particular, $S({}_RRg_i) = S(e_{i1}R_R)$ is a simple ideal. In the matrix representation,

$$S_k({}_RRg_i) = \begin{pmatrix} & 0 & \\ & \vdots & \\ & 0 & \\ & X_1 & \\ 0 & \vdots & 0 \\ & X_k & \\ & 0 & \\ & \vdots & \\ & 0 & \end{pmatrix}$$

where $X_j = S({}_{e_{ij}Re_{ij}}e_{ij}Rg_i) = S(e_{ij}Rg_{ig_iRg_i})$ for $1 \leq j \leq k$.

Here we define two mappings :

$$\sigma: \{1,\dots, m\} \to \{1,\dots, m\}$$
$$\rho: \{1,\dots, m\} \to \{1,\dots, n(1)\} \cup \cdots \cup \{1,\dots, n(m)\}$$

by the rule $\sigma(i) = k$ and $\rho(i) = t$ if $g_i = e_{kt}$; namely $(e_{i1}R_R\,;\ {}_RRe_{\sigma(i)\rho(i)})$ is an injective pair. We note that $\{\sigma(1),\dots, \sigma(m)\} \subseteq \{1,\dots, m\}$ and $\rho(i) \leq n(\sigma(i))$.

We define R_{ij} by putting

$$R_{ij} = \begin{pmatrix} (e_{j1}, e_{i1}) & \cdots & (e_{jn(j)}, e_{i1}) \\ & \cdots & \\ (e_{j1}, e_{in(i)}) & \cdots & (e_{jn(j)}, e_{in(i)}) \end{pmatrix} = \begin{pmatrix} e_{i1}Re_{j1} & \cdots & e_{i1}Re_{jn(j)} \\ & \cdots & \\ e_{in(i)}Re_{j1} & \cdots & e_{in(i)}Re_{jn(j)} \end{pmatrix}$$

Then

$$R = \begin{pmatrix} R_{11} & \cdots & R_{1m} \\ & \cdots & \\ R_{m1} & \cdots & R_{mm} \end{pmatrix}$$

We define $P_{ik,jt}$ corresponding to $e_{ik}Re_{jt}$ as follows :

$$P_{ik,jt} = \begin{cases} e_{i1}Re_{j1} = (e_{j1}, e_{i1}) & (i \neq j) \\ e_{i1}Re_{j1} = (e_{j1}, e_{i1}) & (i = j,\ k \leq t) \\ J(e_{ij}Re_{j1}) = J((e_{j1}, e_{i1})) & (i = j,\ k > t) \end{cases}$$

and put

$$P_{ij} = \begin{pmatrix} P_{i1,j1} & \cdots & P_{i1,jn(j)} \\ & \cdots & \\ P_{in(i),j1} & \cdots & P_{in(i),jn(j)} \end{pmatrix}$$

Namely, when $i \neq j$,

$$P_{ij} = \begin{pmatrix} e_{i1}Re_{i1} & \cdots & e_{i1}Re_{i1} \\ & \cdots & \\ e_{i1}Re_{i1} & \cdots & e_{i1}Re_{i1} \end{pmatrix}$$

and when $i = j$,

$$P_{ij}=P_{ii}=\begin{pmatrix} e_{i1}Re_{i1} & \cdots & e_{i1}Re_{i1} \\ & \ddots & \\ J(e_{i1}Re_{i1}) & \cdots & e_{i1}Re_{i1} \end{pmatrix}$$

We put

$$P=\begin{pmatrix} P_{11} & \cdots & P_{1m} \\ & \cdots & \\ P_{m1} & \cdots & P_{mm} \end{pmatrix}$$

Then P becomes a ring by usual matrix operations. Let p_{ij} be the element of P such that its (ij, ij) position is the unity of $P_{ij,ij}$ and all other positions are zero. Then $\{p_{11},\dots, p_{1n(1)},\dots, p_{m1},\dots, p_{mn(m)}\}$ is a complete set of orthogonal primitive idempotents of P; $P=p_{11}P\oplus\cdots\oplus p_{1n(1)}P\oplus\cdots\oplus p_{m1}P\oplus\cdots\oplus p_{mn(m)}P$.

For each (ik, jt), we define a mapping

$$\begin{array}{ccc} \tau_{ik,jt}:\ P_{ik,jt} & \to & e_{ik}Re_{jt} \\ \cap\!| & & \| \\ (e_{j1}, e_{i1}) & & (e_{jt}, e_{ik}) \end{array}$$

by the rule: $\alpha\to(\theta^i_{k,1})^{-1}\alpha\theta^j_{t,1}$. Of course, the mapping is well defined. In fact, let $\alpha\in(e_{j1}, e_{i1})$. If $i\neq j$, $\operatorname{Im}\alpha\subseteq J_{n(i)}(e_{i1}R)\subset J_{k-1}(e_{i1}R)=\operatorname{Im}\theta^i_{k,1}$ (Note that R is a basic ring). If $i=j$ and $k\leq t$, then, $\operatorname{Im}\alpha\theta^i_{t,1}\subseteq J_{t-1}(e_{i1}R)\subseteq J_{k-1}(e_{i1}R)=\operatorname{Im}\theta^i_{k-1}$. If $i=j$ and $k>t$, then as $\alpha\in J(e_{i1}Re_{i1})$, α is non unit, so $\operatorname{Im}\alpha\subseteq J_{n(i)}(e_{i1}R)\subseteq J_{k-1}(e_{i1}R)=\operatorname{Im}\theta^i_{k,1}$. Thus $\tau_{ik,jt}$ is well defined. Note that $\tau_{ik,jt}$ is a group homomorphism.

We need the following proposition

Proposition 1. *Each $\tau_{ik,jt}$ is an epimorphism and $\tau_{ik,jt}\tau_{jt,pq}=\tau_{ik,pq}$ for each (ik, jt) and (jt, pq).*

Proof. Let $\beta\in(e_{jt}, e_{ik})$. When $i\neq j$, consider the following diagram:

$$\begin{array}{ccccc} 0 \to & e_{jt}R & \xrightarrow{\theta^j_{t,1}} & J_{t-1}(e_{j1}R) & \subseteq e_{j1}R \\ & \downarrow\beta & & & \\ & e_{ik}R & & & \\ & \downarrow\theta^i_{k,1} & & & \\ & J_{k-1}(e_{i1}R) & & & \\ & \cap\!| & & & \\ & e_{i1}R & & & \end{array}$$

Since $e_{i1}R_R$ is injective, there exists $\alpha \in (e_{j1}, e_{i1})$ such that $\alpha\theta^i_{t,1} = \theta^i_{k,1}\beta$. Noting $i \neq j$, we see $\operatorname{Im}\alpha \subseteq J_{n(i)}(e_{i1}R)$; so $\operatorname{Im}\alpha\theta^i_{t,1} \subseteq J_{k-1}(e_{i1}R)$ and hence we see $\beta = (\theta^i_{k,1})^{-1}\alpha\theta^j_{t,1} = \tau_{ik,jt}(\alpha)$.

When $i = j$ and $k \leq t$, consider the following

$$\begin{array}{l} 0 \to e_{it}R \xrightarrow{\theta^i_{t,1}} J_{t-1}(e_{i1}R) \subseteq e_{i1}R \\ \qquad\ \downarrow \beta \\ \qquad\ e_{ik}R \\ \qquad\ \ \downarrow \theta^i_{k,1} \\ \quad J_{k-1}(e_{i1}R) \\ \qquad\quad \cap\!| \\ \qquad\ e_{i1}R \end{array}$$

Then there exists $\alpha \in (e_{i1}, e_{i1})$ satisfying $\alpha\theta^i_{t,1} = \theta^i_{k,1}\beta$. Since $k \leq t$, $\operatorname{Im}\alpha\theta^i_{t,1} \subseteq J_{t-1}(e_{i1}R) \subseteq J_{k-1}(e_{i1}R)$. Hence $\beta = (\theta^i_{k,1})^{-1}\alpha\theta^i_{t,1} = \tau_{ik,it}(\alpha)$.

Next, when $i = j$ and $k > t$, again consider the diagram

$$\begin{array}{l} 0 \to e_{it}R \xrightarrow{\theta^i_{t,1}} J_{t-1}(e_{i1}R) \subseteq e_{i1}R \\ \qquad\ \downarrow \beta \\ \qquad\ e_{ik}R \\ \qquad\ \ \downarrow \theta^i_{k,1} \\ \quad J_{k-1}(e_{i1}R) \\ \qquad\quad \cap\!| \\ \qquad\ e_{i1}R \end{array}$$

We take $\alpha \in (e_{i1}, e_{i1})$ satisfying $\alpha\theta^i_{t,1} = \theta^i_{k,1}\beta$. If α is an isomorphism, we see from $k > t$ that $\operatorname{Im}\alpha\theta^i_{t,1} = J_{t-1}(e_{i1}R) \supsetneqq J_{k-1}(e_{i1}R) \supseteq \operatorname{Im}\theta^i_{k,1}\beta$, which contracts $\alpha\theta^i_{t,1} = \theta^i_{k,1}\beta$. Hence α must be non-unit, so $\alpha \in J((e_{i1}, e_{i1}))$. Since $\operatorname{Im}\alpha \subseteq J_{n(i)}(e_{i1}\theta)$, $\operatorname{Im}\alpha\theta^i_{t,1} \subseteq J_{k-1}(e_{i1}R)$. Hence $\beta = (\theta^i_{k,1})^{-1}\alpha\theta^i_{t,1} = \tau_{ik,it}(\alpha)$. Thus we see that $\tau_{ik,jt}$ is an epimorphism.

For $\alpha \in P_{jt,pq}$ and $\beta \in P_{ik,jt}$, $\tau_{ik,jt}(\beta)\tau_{jt,pq}(\alpha) = ((\theta^i_{k,1})^{-1}\beta\theta^j_{t,1})((\theta^j_{t,1})^{-1}\cdot \alpha\theta^p_{q,1}) = (\theta^i_{k,1})^{-1}\beta\alpha\theta^p_{q,1}$; whence $\tau_{ij,kt}\tau_{jt,pq} = \tau_{ik,pq}$.

By Proposition 1, we see that

$$\tau = \begin{pmatrix} \tau_{11,11} & \cdots & \tau_{11,mn(m)} \\ & \cdots & \\ \tau_{mn(m),11} & \cdots & \tau_{mn(m),mn(m)} \end{pmatrix}$$

gives a ring epimorphism from P to R with $\tau(p_{ik}) = e_{ik}$ for all ik.

Proposition 2. 1) *If $j \neq \sigma(i)$, then $\tau_{ik,jt}$ is an isomorphism.*

2) *If $j = \sigma(i)$ and $t \leq \rho(i)$, then $\tau_{ik,jt}$ is also an isomorphism.*

3) *If $j = \sigma(i)$ and $t > \rho(i)$, then $\tau_{ik,jt}$ is not an isomorphism. In fact,*

$$\begin{aligned}\operatorname{Ker} \tau_{ik,\sigma(i)t} &= \{\alpha \in P_{ik,\sigma(i)t} \mid \operatorname{Ker} \alpha = J_{\rho(i)}(e_{\sigma(i)1} R)\} \cup 0 \\ &= S({}_{e_{i1}Re_{i1}} e_{i1} R e_{\sigma(i)1}) \\ &= S({}_{P_{ik,ik}} P_{ik,\sigma(i)t}) \\ &= S(e_{i1} R e_{\sigma(i)1} {}_{e_{\sigma(i)1}Re_{\sigma(i)1}}) \\ &= S(P_{ik,\sigma(i)t} {}_{P_{\sigma(i)t,\sigma(i)t}})\end{aligned}$$

So, in this case, $\operatorname{Ker} \tau_{ik,jt}$ *is a simple module as a left $P_{ik,ik}$- and a right $P_{jt,jt}$-module.*

Proof. We can take an epimorphism β of $e_{\sigma(i)\rho(i)}R_R$ to $S(e_{i1}R)_R$.

1) Assume $j \neq \sigma(i)$. If there exists a non-zero $\alpha \in \rho_{ik,jt}$ such that $0 = \tau_{ik,jt}(\alpha) = (\theta^i_{k,1})^{-1}\alpha\theta^j_{t,1}$, then $\alpha\theta^j_{t,1} = 0$; so $\operatorname{Ker} \alpha \supseteq J_{t-1}(e_{j1}R)$. Consider the diagram

$$\begin{array}{c} e_{\sigma(i)\rho(i)}R \\ \downarrow \beta \\ S(e_{i1}R) \\ \cap\!\mid \\ e_{j1}R \xrightarrow{\alpha} \operatorname{Im} \alpha \to 0 \end{array}$$

Since $e_{\sigma(i)\rho(i)}R_R$ is projective, there exists γ such that $\alpha\gamma = \beta$. Since $j \neq \sigma(i)$ and R is basic, γ is not an epimorphism ; whence $\operatorname{Im} \gamma \subseteq J_{n(j)}(e_{j1}R) \subseteq J_{t-1}(e_{j1}R) \subseteq \operatorname{Ker} \alpha$ and hence $\beta = \alpha\gamma = 0$, a contradiction. Thus, $\operatorname{Ker} \tau_{ik,jt} = 0$ and hence $\tau_{ik,jt}$ is an isomorphism (cf. Proposition 1).

2) Assume $j = \sigma(i)$ and $t \leq \rho(i)$. If there exists $\alpha \in P_{ik,jt}$ such that $0 = \tau_{ik,\sigma(i)t}(\alpha) = (\theta^i_{k,1})^{-1}\alpha\theta^{\sigma(i)}_{t,1}$, then $\alpha\theta^{\sigma(i)}_{t,1} = 0$; so $\operatorname{Ker} \alpha \supseteq J_{t-1}(e_{\sigma(i)1}R)$. Consider the diagram :

$$\begin{array}{c} e_{\sigma(i)\rho(i)}R \\ \downarrow \beta \\ S(e_{i1}R) \\ \cap\!\mid \\ e_{\sigma(i)1}R \xrightarrow{\alpha} \operatorname{Im} \alpha \to 0 \end{array}$$

Since $e_{\sigma(i)\rho(i)}R_R$ is projective, there exists $\gamma \in (e_{\sigma(i)\rho(i)}, e_{\sigma(i)1})$ such that $\alpha\gamma = \beta$. If $\rho(i) = 1$, then $t = 1$ since $\rho(i) \geq t$; which implies $\operatorname{Ker} \alpha \supseteq$

$J_0(e_{\sigma(i)1}R) = e_{\sigma(i)1}R$; so $\alpha = 0$, a contradiction. Hence $\rho(i) \neq 1$. As γ is not an epimorphism (R is basic), $\mathrm{Im}\,\gamma \subseteq J_{\rho(i)-1}(e_{\sigma(i)1}R)$. Since $t \leq \rho(i)$, $J_{\rho(i)-1}(e_{\sigma(i)1}R) \subseteq J_t(e_{\sigma(i)1}R) \subseteq \mathrm{Ker}\,\alpha$; whence $\beta = \alpha\gamma = 0$, a contradiction. Therefore $\mathrm{Ker}\,\tau_{ik,\sigma(i)t} = 0$ and so $\tau_{ik,\sigma(i)t}$ is an isomorphism.

3) Assume $j = \sigma(i)$ and $t > \rho(i)$. We put $X = \{\alpha \in P_{ik,\sigma(i)t} \mid \mathrm{Ker}\,\alpha = J_{\rho(i)}(e_{\sigma(i)1}R)\} \cup 0$ and we first show that $\mathrm{Ker}\,\tau_{ik,\sigma(i)t} = X$. Let $0 \neq \alpha \in \mathrm{Ker}\,\tau_{ik,\sigma(i)t}$. Then since $0 = \tau_{ik,\sigma(i)t}(\alpha) = (\theta^i_{k,1})^{-1}\alpha\theta^{\sigma(i)}_{t,1}$, $\mathrm{Ker}\,\alpha \supseteq J_{t-1}(e_{\sigma(i)1}R)$ (Note that $t \geq 2$ since $t > \rho(i)$). Hence $\mathrm{Ker}\,\alpha = J_{s-1}(e_{\sigma(i)1}R)$ for some $s \leq t$. We claim that $s-1 = \rho(i)$. To see this, let $\bar{\alpha}$ be the restriction of α to $J_{s-2}(e_{\sigma(i)1}R)$. Then $\mathrm{Ker}\,\bar{\alpha} = \mathrm{Ker}\,\alpha$ and $\mathrm{Im}\,\bar{\alpha} \simeq J_{s-2}(e_{\sigma(i)1}R)/\mathrm{Ker}\,\bar{\alpha} = J_{s-2}(e_{\sigma(i)1}R)/J_{s-1}(e_{\sigma(i)1}R)$; so $\mathrm{Im}\,\bar{\alpha} = S(e_{i1}R)$. Therefore $\bar{\alpha}\theta^{\sigma(i)}_{s-1,1}$ is an epimorphism of $e_{\sigma(i),s-1}R$ to $S(e_{i1}R)$. Hence $s-1 = \rho(i)$ and therefore $\mathrm{Ker}\,\alpha = J_{\rho(i)}(e_{\sigma(i)1}R)$. Thus $\alpha \in X$; whence $\mathrm{Ker}\,\tau_{ik,\sigma(i)t} \subseteq X$.

Conversely, let $0 \neq \alpha \in X$. Since $\mathrm{Ker}\,\alpha = J_{\rho(i)}(e_{\sigma(i)1}R)$ and $\rho(i) \leq t-1$, $\mathrm{Im}\,\theta^{\sigma(i)}_{t,1} = J_{t-1}(e_{\sigma(i)1}R) \subseteq J_{\rho(i)}(e_{\sigma(i)1}R) = \mathrm{Ker}\,\alpha$; whence $0 = (\theta^i_{k,1})^{-1}\alpha\theta^{\sigma(i)}_{t,1} = \tau_{ik,\sigma(i)t}(\alpha)$ and $\alpha \in \mathrm{Ker}\,\tau_{ik,\sigma(i)t}$. Thus $X \subseteq \mathrm{Ker}\,\tau_{ik,\sigma(i)t}$ and hence $X = \mathrm{Ker}\,\tau_{ik,\sigma(i)t}$.

Next, we show that $\mathrm{Ker}\,\tau_{ik,\sigma(i)t} \neq 0$. Consider the diagram:

$$
\begin{array}{l}
0 \to e_{\sigma(i)\rho(i)}R \xrightarrow{\theta^{\sigma(i)}_{\rho(i),1}} J_{\rho(i)-1}(e_{\sigma(i)1}R) \subseteq e_{\sigma(i)1}R \\
\qquad\quad \downarrow\beta \\
\qquad\quad S(e_{i1}R) \\
\qquad\qquad \cap \\
\qquad\qquad e_{i1}R
\end{array}
$$

Since $e_{i1}R_R$ is injective, there exists non-zero $\alpha \in P_{ik,\sigma(i)t}$ such that $\beta = \alpha\theta^{\sigma(i)}_{\rho(i),1}$. We claim $\alpha \in \mathrm{Ker}\,\tau_{ik,\sigma(i)t}$, i.e., $\mathrm{Ker}\,\alpha = J_{\rho(i)}(e_{\sigma(i)1}R)$. If $x \in J_{\rho(i)}(e_{\sigma(i)1}R)$, then we see from $(\theta^{\sigma(i)}_{\rho(i)1})^{-1}(x) \in J(e_{\sigma(i)\rho(i)}R) = \mathrm{Ker}\,\beta$ that $\beta((\theta^{\sigma(i)}_{\rho(i),1})^{-1}(x) = \alpha\theta^{\sigma(i)}_{\rho(i),1})(\theta^{\sigma(i)}_{\rho(i),1})^{-1}(x) = \alpha(x) = 0$; so $x \in \mathrm{Ker}\,\alpha$. Hence $J_{\rho(i)}(e_{\sigma(i)1}R) \subseteq \mathrm{Ker}\,\alpha$. If $\mathrm{Ker}\,\alpha \supseteq J_f(e_{\sigma(i)1}R)$ for some $f < \rho(i)$, then $\beta(e_{\sigma(i)1}R) = \alpha\theta^{\sigma(i)}_{\rho(i),1}(e_{\sigma(i)1}R) = \alpha(J_{\rho(i)-1}(e_{\sigma(i)1}R) = 0$ and hence $\beta = 0$, a contradiction. Thus we see $\mathrm{Ker}\,\alpha \subseteq J_{\rho(i)}(e_{\sigma(i)1}R)$ and $\mathrm{Ker}\,\alpha = J_{\rho(i)}(e_{\sigma(i)1}R)$.

To show the remainder, we put $\psi = \tau_{i1,\sigma(i)\rho(i)}$; so ψ is an isomorphism of $e_{i1}Re_{\sigma(i)1}$ to $e_{i1}Re_{\sigma(i)\rho(i)}$ by the rule $\psi(\alpha) = (\theta^i_{1,1})^{-1}\alpha\theta^{\sigma(i)}_{\rho(i),1} = \alpha\theta^{\sigma(i)}_{\rho(i),1}$, and put $Y = \{\alpha \in (e_{\sigma(i)\rho(i)}, e_{i1}) \mid \mathrm{Im}\,\alpha \subseteq S(e_{i1}R)\}$. We see from [4] that $0 \neq Y = S(e_{i1}Re_{\sigma(i)\rho(i)e_{\sigma(i)\rho(i)}Re_{\sigma(i)\rho(i)}}) = S(_{e_{i1}Re_{i1}}e_{i1}Re_{\sigma(i)\rho(i)})$. We show $\psi(X) = Y$. Let $0 \neq \alpha \in X$. If $\mathrm{Ker}\,\alpha\theta^{\sigma(i)}_{\rho(i),1} = e_{\sigma(i)\rho(i)}R$, then $\psi(\alpha) = \alpha\theta^{\sigma(i)}_{\rho(i),1} = 0$, so $\alpha = 0$, a contradiction. Therefore $\mathrm{Ker}\,\alpha\theta^{\sigma(i)}_{\rho(i)} \subseteq J(e_{\sigma(i)\rho(i)}R)$. On the other hand, for any $x \in J(e_{\sigma(i)\rho(i)}R)$, $\theta^{\sigma(i)}_{\rho(i),1}(x) \in J(J_{\rho(i)-1}(e_{\sigma(i)1}R)) = J_{\rho(i)}(e_{\sigma(i)1}R) = \operatorname{Ker}\alpha$; so $\alpha\theta^{\sigma(i)}_{\rho(i),1}(x) = 0$ and hence $x \in \operatorname{Ker}\alpha\theta^{\sigma(i)}_{\rho(i),1}$. Thus $J(e_{\sigma(i)\rho(i)}R) \subseteq \operatorname{Ker}\alpha\theta^{\sigma(i)}_{\rho(i)}$ and hence $J(e_{\sigma(i)\rho(i)}R) = \operatorname{Ker}\alpha\theta^{\sigma(i)}_{\rho(i)}$, from which we see that $\operatorname{Im}\alpha\theta^{\sigma(i)}_{\rho(i),1} = S(e_{i1}R_R)$ and $\psi(\alpha) \in Y$; as a result, $\psi(X) \subseteq Y$. Conversely, for any non-zero $\alpha \in Y$, we can take $\beta \in e_{i1}Re_{\sigma(i)1}$ such that $\psi(\beta) = \alpha$, since ψ is an isomorphism. If $\operatorname{Ker}\beta \supseteq J_t(e_{\sigma(i)1}R)$ for some $t < \rho(i)$, then $\alpha = \psi(\beta) = \beta\theta^{\sigma(i)}_{\rho(i),1} = \beta(J_{\rho(i)-1}(e_{\sigma(i)}R)) = 0$, a contradiction. Hence $\operatorname{Ker}\beta \subseteq J_{\rho(i)}(e_{\sigma(i)1}R)$. On the other hand, for any $x \in J_{\rho(i)}(e_{\sigma(i)1}R)$, we see from $(\theta^{\sigma(i)}_{\rho(i),1})^{-1}(x) \in J(e_{\sigma(i)\rho(i)}R)$ that $\alpha((\theta^{\sigma(i)}_{\rho(i),1})^{-1}(x)) = 0$. Since $\beta = \alpha\theta^{\sigma(i)}_{\rho(i),1}$, it follows that $\beta(x) = 0$; whence $x \in \operatorname{Ker}\beta$. Thus $\operatorname{Ker}\beta = J_{\rho(i)}(e_{\sigma(i)1}R)$ and hence $\beta \in X$ and we obtain $Y \subseteq \psi(X)$.

To show $\psi(X) \subseteq Y$, we put $\mu = \tau_{\sigma(i)\rho(i),\sigma(i)\rho(i)}$, i.e., μ is the mapping of $e_{\sigma(i)1}Re_{\sigma(i)1}$ to $e_{\sigma(i)\rho(i)}Re_{\sigma(i)\rho(i)}$ given by $\mu(\alpha) = (\theta^{\sigma(i)}_{\rho(i),1})^{-1}\alpha\theta^{\sigma(i)}_{\rho(i),1}$. As we saw above, μ is an group isomorphism. Moreover, this map is a ring isomorphism. By this isomorphism, $e_{i1}Re_{\sigma(i)\rho(i)}$ becomes a right $e_{\sigma(i)1}Re_{\sigma(i)1}$-module. For $\alpha \in e_{i1}Re_{\sigma(i)1}$ and $x \in e_{\sigma(i)1}Re_{\sigma(i)1}$, $\psi(\alpha x) = \alpha x\theta^{\sigma(i)}_{\rho(i),1} = (\alpha\theta^{\sigma(i)}_{\rho(i),1})\cdot((\theta^{\sigma(i)}_{\rho(i),1})^{-1}x\theta^{\sigma(i)}_{\rho(i),1}) = \psi(\alpha)\mu(x) = \psi(\alpha)x$. Hence ψ is an isomorphism between right $e_{\sigma(i)1}Re_{\sigma(i)1}$-modules. Since $Y = S(e_{i1}Re_{\sigma(i)\rho(i)e_{\sigma(i)\rho(i)}Re_{\sigma(i)\rho(i)}})$, it follows that $X = S(e_{i1}Re_{\sigma(i)1 e_{\sigma(i)1}Re_{\sigma(i)1}}) = S(P_{ik,\sigma(i)tP_{\sigma(i)t,\sigma(i)t}})$. Finally, we see from $Y = S(_{e_{i1}Re_{i1}}e_{i1}Re_{\sigma(i)\rho(i)})$ and $_{e_{i1}Re_{i1}}e_{i1}Re_{i1} \simeq {}_{e_{i1}Re_{i1}}e_{i1}Re_{\sigma(i)1}$ that $\operatorname{Ker}\tau_{ik,\sigma(i)t} = S(_{P_{ik,ik}}P_{ik,\sigma(i)t})$.

We replace $P_{ik,\sigma(i)t}$ in

$$P_{i\sigma(i)} = \begin{pmatrix} P_{i1,\sigma(i)1} & \cdots & P_{i1,\sigma(i)n(\sigma(i))} \\ & \cdots & \\ P_{in(i),\sigma(i)1} & \cdots & P_{in(i),\sigma(i)n(\sigma(i))} \end{pmatrix}$$

by $P_{ik,\sigma(i)t}/S(P_{ik,\sigma(i)t})$ for $k = 1,\dots, n(i)$ and $j = \rho(i)+1,\dots, n(\sigma(i))$, and denote it by $P^*_{i\sigma(i)}$. And we put

$$R^* = \begin{pmatrix} P_{11}\cdots P_{1,\sigma(i)-1}P^*_{1\sigma(1)}P_{1,\sigma(i)+1}\cdots P_{1m} \\ \cdots \\ P_{m1}\cdots P_{m,\sigma(m)-1}P^*_{m\sigma(m)}P_{m,\sigma(m)+1}\cdots P_{mn} \end{pmatrix}$$

Then by Propositions 1 and 2, we obtain

Theorem 1. *R^* canonically becomes a ring such that $R^* \simeq P/\operatorname{Ker}\tau \simeq R$. Therefore R^* is a representative matrix ring of R.*

We shall illustrate this theorem for two cases :

1) The case : $m=1$. Then

$$R \simeq R^{*}=\left(\begin{array}{ccccccc} \overbrace{Q \cdots\cdots Q}^{\rho(1)} & \bar{Q} & \cdots & \bar{Q} \\ J \ \ddots \quad \vdots & \vdots & & \vdots \\ \vdots \quad \ddots \quad Q & \vdots & & \vdots \\ \vdots \qquad\quad J & \bar{Q} & & \vdots \\ \vdots \qquad\quad \vdots & \bar{J} & \ddots & \vdots \\ \vdots \qquad\quad \vdots & \vdots & \ddots & \vdots \\ J \cdots\cdots J & \bar{J} & \cdots \ \bar{J} & \bar{Q} \end{array}\right)$$

where $Q=e_{11}Re_{11}$, $J=J(Q)$, $\bar{Q}=Q/S(Q)$. (In this case, Q is a local QF-ring, since both ${}_{Q}S({}_{Q}Q)$ and $S(Q_{Q})_{Q}$ are simple (cf. [5])).

2) The case : $m=2$, $\sigma(1)=2$ and $\sigma(2)=1$. Then

$$R \simeq R^{*}=\left(\begin{array}{ccc|ccc} Q \cdots\cdots & & Q & \overbrace{A \cdots A}^{\rho(1)} & \bar{A} \cdots & \bar{A} \\ & \ddots & \vdots & \vdots \quad \vdots & \vdots & \vdots \\ J & & Q & A \cdots A & \bar{A} \cdots & \bar{A} \\ B \cdots B & \bar{B} \cdots & \bar{B} & T \cdots\cdots & & T \\ \vdots \quad \vdots & \vdots & \vdots & & \ddots & \vdots \\ \underbrace{B \cdots B}_{\rho(2)} & \bar{B} \cdots & \bar{B} & K & & T \end{array}\right)$$

where $Q=e_{11}Re_{11}$, $T=e_{21}Re_{21}$, $A=e_{11}Re_{21}$, $\bar{A}=A/S(A)$, $B=e_{21}Re_{11}$, $\bar{B}=B/S(B)$, $J=J(Q)$ and $K=J(T)$. (Then $T=\begin{pmatrix} Q & A \\ B & T \end{pmatrix}$ is a QF-ring, since $(eT_{T}\,;\,{}_{T}Tf)$ and $(fT_{T}\,;\,{}_{T}Te)$ are injective pairs, where $e=\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $f=\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$)

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