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# ON HARADA RINGS. I

To Takasi Nagahara on his 60th birthday

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In [4]−[6] (cf. [7]), M. Harada has studied the following two conditions:

(∗) Every non-small right $R$-module contains a non-zero injective submodule.

(∗)* Every non-cosmall right $R$-module contains a non-zero projective summand.

In particular, he has studied two rings which are characterized by ideal theoretic conditions; the one is a perfect ring with (∗) and the other is a semi-perfect ring with (∗)*. In [8] and [9], the author has studied these rings with some additional conditions and introduced Harada (abbreviated $H$-) rings and co-Harada rings: A ring $R$ is a right $H$-ring if it is a right artinian ring with (∗), and dually $R$ is a right co-$H$-ring if it satisfies (∗)* and the ascending chain condition for right annihilator ideals. In view of results in there, we see that these two rings are 'companions' of $QF$-rings and generalized uniserial rings. Although these classical artinian rings are left-right symmetric, $H$- and co-$H$-rings are not left-right symmetric. This fact seems to be an interesting phenomenon. However, in the present paper, we shall show a more interesting fact that the left $H$-rings and the right co-$H$-rings are the same rings. As a by-product of the study of $H$-rings, we shall show that a right generalized right $QF$-3 ring is a generalized uniserial ring.

**1. Preliminaries.** Throughout this paper, we assume that all rings $R$ considered are associative rings with identity, all $R$-modules are unitary and all homomorphisms between $R$-modules are written on the opposite side of scalars. The notation $M_R$ (resp. $_RM$) is used to denote that $M$ is a right (resp. left) $R$-modules. Let $M$ be an $R$-module. We use $E(M)$, $J(M)$, $S(M)$ and $Z(M)$ to denote its injective hull, Jacobson radical, socle and singular submodule, respectively. Further, by $\{J_i(M)\}$ and $\{S_i(M)\}$, we denote its descending Loewy chain and ascending Loewy chain, respectively, i.e., $J_0(M) = M$, $J_1(M) = J(M)$, $J_2(M) = J(J_1(M)), \dots$, $S_0(M) = 0$, $S_1(M) = S(M)$, $S_2(M)/S_1(M) = S(M/S_1(M)), \dots$.

For submodules $A$ and $B$ of an $R$-module $M$ with $A \subseteq B$, the notation $A \subseteq_e B$ stands for '$A$ is an essential submodule of $B$' ; while $A \subseteq_c B$ (in $M$) means '$A$ is a co-essential submodule of $B$', i.e., $B/A$ is a small submodule of $M/A$. For two R-modules $M$ and $N$, we use $M \subsetneqq N$ to stand for there is a monomorphism $f$ from $M$ into $N$; in particular, $M \subsetneqq_e N$ means that such $f$ exists and $f(M) \subseteq_e N$. The term $ACC$ means the ascending chain condition.

**Definition.** An $R$-module $M$ is an *extending* (resp. *lifting*) module if, for any submodule $A$ of $M$, there exists a direct summand $A^*$ of $M$ such that $A \subseteq_e A^*$ (resp. $A^* \subseteq_c A$).

**Definition** ([4]−[7], [10]). An $R$-module $M$ is a small module if it is small in its injective hull, and $M$ is a non-small module if it is not a small module. Dually, $M$ is a cosmall module if, for any projective module $P$ and any epimorphism $f$: $P \to M$, ker $f$ is an essential submodule of $P$, and $M$ is a non-cosmall module if it is not a cosmall module.

**Definition** ([1], [2], [7], [11]). A ring $R$ is said to be right $QF$-3 if it has a minimal faithful right ideal. Right and left $QF$-3 rings are said to be $QF$-3 rings.

The following lemma due to Fuller plays an important role in our study.

**Lemma 1.1.** *Let $R$ be a left artinian ring and $f$ a primitive idempotent of $R$. Then ${}_RRf$ is injective if and only if there exists a primitive idempotent $e$ in $R$ such that $(eR_R\,;\,{}_RRf)$ is an injective pair, i.e.,*

$$ {}_RRe/J({}_RRe) \simeq {}_RS({}_RRf) \text{ and } S(eR_R)_R \simeq fR_R/J(fR_R) $$

*Moreover, when this is so, $eR_R$ is also injective.*

**Notation.** Let $R$ be a left artinian ring with a complete set of orthogonal primitive idempotents. When $E$ is arranged as $E = \{e_1, e_2, \ldots, e_n\}$, we can identify $R$ with the matrix ring:

$$ \begin{bmatrix} e_1Re_1 & \cdots & e_1Re_n \\ & \cdots & \\ e_nRe_1 & \cdots & e_nRe_n \end{bmatrix} $$

For the sake of convenience, we use the terms $e_i$-row and $e_i$-column instead of the terms $i$-row and $i$-column, respectively. So we identify $e_iR$ and $Re_i$ with $e_i$-row and $e_i$-column, respectively.

We note that if $R$ is basic and $(e_iR_R\,;\,{}_RRe_j)$ is an injective pair, then

$$S(e_iR_R)=\begin{bmatrix} & & & 0 & & & \\ & 0 & & \vdots & & 0 & \\ & & & 0 & & & \\ 0 & \cdots & 0 & S(e_iRe_{j\,e_jRe_j}) & 0 & \cdots & 0 \\ & & & 0 & & & \\ & 0 & & \vdots & & 0 & \\ & & & 0 & & & \end{bmatrix}$$

$$S({}_RRe_j)=\begin{bmatrix} & & & 0 & & & \\ & 0 & & \vdots & & 0 & \\ & & & 0 & & & \\ 0 & \cdots & 0 & S({}_{e_iRe_i}e_iRe_j) & 0 & \cdots & 0 \\ & & & 0 & & & \\ & 0 & & \vdots & & 0 & \\ & & & 0 & & & \end{bmatrix}$$

**Lemma 1.2** (Rayer [10], cf. [6]). *Let $R$ be a right artinian ring, and let $M$ be a right $R$-module. Then $M$ is a small module if and only if* $MS({}_RR)=0$.

2. **Background.** As mentioned in the introduction, Harada has studied the conditions: (*) Every non-small right $R$-module contains a non-zero injective submodule. (*)* Every non-cosmall right $R$-module contains a non-zero projective summand. And he has shown the following two theorems which give ideal theoretic characterizations of right artinian rings with (*) and semi-perfect rings with (*)*.

**Theorem 2.1** ([6, Theorem 2,3]). *A right artinian ring $R$ satisfies (*) if and only if, for any primitive idempotent $e$ in $R$ such that $eR_R$ is non-small, there exists an integer $t\geqq 0$ for which*

*a) $eR_R/S_K(eR_R)$ is injective for $0\leqq k\leqq t$, and*

*b) $eR_R/S_{t+1}(eR_R)$ is a small module.*

**Remark.** A left and right perfect ring with (*) is right artinian ([4, Theorem 5]).

**Theorem 2.2** ([5], [6]). *A semi-perfect R ring satisfies* $(*)^*$ *if and only if, for a complete set* $\{e_i\} \cup \{f_j\}$ *of orthogonal primitive idempotents of R such that each* $e_iR_R$ *is non-small and each* $f_jR_R$ *is small,*

a) *each* $e_iR_R$ *is injective,*

b) *for each* $e_iR$, *there exists* $t_i \geqq 0$ *such that* $J_t(e_iR_R)_R$ *is projective for* $0 \leqq t \leqq t_i$ *and* $J_{t_i-1}(e_iR_R)_R$ *is a singular module, and*

c) *for each* $f_jR$, *there exists* $e_i$ *such that* $f_jR_R \subsetneq e_iR_R$.

**Remark.** In case $R$ is left or right artinian, the condition '$J_{t_i+1}(e_iR_R)$ is singular' in b) above can be dropped as we see in section 4.

**Definition.** We call a ring $R$ a right Harada (abbreviated $H$-) ring if $R$ is a right artinian ring with $(*)$, and call $R$ a right co-Harada ring if $R$ satisfies $(*)^*$ and the $ACC$ for right annihilator ideals. Left and right $H$- (resp. co-$H$-) rings are called $H$- (resp. co-$H$-) rings.

Of course, right $H$-ring and right co-$H$-rings are mutually dual notions, as the following theorems shows:

**Theorem 2.3** ([8]). *The following conditions are equivalent for a given ring R:*

1) *R is a right H-ring.*

2) *Every injective right R-module is a lifting module.*

3) *R is a right perfect ring with the property that the family of all injective right R-modules is closed under taking small covers.*

4) *Every right R-module is expressed as a direct sum of an injective module and a small module.*

*When this is so, R is a QF-3 ring and satisfies the ACC on left annihilator ideals (cf. [1]).*

**Theorem 2.4** ([8]). *The following conditions are equivalent for a given ring R:*

1) *R is a right co-H-ring.*

2) *Every projective right R-module is an extending module.*

3) *The family of all projective right R-modules is closed under taking essential extensions.*

4) *Every right R-module is expressed as a direct sum of a projective*

*module and a singular module.*

*When this is so, $R$ is semi-primary QF-3 and satisfies the ACC on left annihilator ideals.*

**Remark.** 1) Right $H$-rings and right co-$H$-rings are Morita invariant. 2) $QF$-rings and generalized uniserial rings are $H$- and co-$H$-rings ([8], [9]). 3) Not all right $H$-rings are left $H$-rings, and not all right co-$H$-rings are left co-$H$-rings ([8]). 4) For a local $QF$-ring $Q$, the following two rings mentioned in [8] are typical examples of right co-$H$-rings and at the same time of left $H$-rings:

$$\begin{bmatrix} Q & Q \\ J & Q \end{bmatrix} \qquad \begin{bmatrix} Q & Q/S \\ J & Q/S \end{bmatrix}$$

where $J = \mathrm{J}(Q)$ and $S = \mathrm{S}(Q)$. 5) For an algebra $R$ over a field of finite dimension, $R$ is a left $H$-ring if and only if it is a right co-$H$-ring.

These remarks 4) and 5) led us to conjecture that left $H$-rings and right co-$H$-rings are the same rings. This conjecture is in fact true. In the next section we prove that right co-$H$-rings are left $H$-rings and, in section 5, the converse.

**3. Right co-H $\Rightarrow$ left H.** In this section, we show that right co-$H$-rings are left $H$-rings. As both rings are Morita invariant, we may show that basic right co-$H$-rings are left $H$-rings. Therefore in this section we assume that $R$ is a basic right co-$H$-ring with a complete set

$$E = \{e_{11}, \dots, e_{1n(1)}, \dots, e_{m1}, \dots, e_{mn(m)}\}$$

of orthogonal primitive idempotents such that

a) each $e_{i1}R_R$ is injective,

b) $e_{i1}R_R \supseteq e_{i2}R_R \supseteq \cdots \supseteq e_{in(i)}R_R$ ; more precisely, there exists an isomorphism $\theta^i_{j,j-1}$ from $e_{ij}R_R$ to $J(e_{i,j-1}R_R)_R$ for $j = 1, \dots, n(i)$ and $i = 1, \dots, m$,

c) all $S(e_{ij}R_R)_R$ are simple.

**Notation.** 1) we put

$$\theta^i_{j,k} = \theta^i_{k+1,k}\,\theta^i_{k+2,k+1} \cdots \theta^i_{j-1,j-2}\,\theta^i_{j,j-1}$$

for $0 \leqq k < j \leqq n(i)$. Then $\theta^i_{j,k}$ is an isomorphism from $e_{ij}R_R$ to $\mathrm{J}_{j-k}(e_{i,k}R_R)_R$.

2) We define a mapping $\Phi^i_{j,j+1}$ from $\mathrm{End}_R(e_{ij}R_R)$ to $\mathrm{End}_R(e_{i,j+1}R_R)$ by the rule

$$\Phi^i_{j,j+1}(\alpha) = (\theta^i_{j+1,j})^{-1}\ \alpha\theta^i_{j+1,j}$$

for $\alpha \in \mathrm{End}_R(e_{ij}R_R)$. Then it is routine to check that $\Phi^i_{j,j+1}$ is a ring epimorphism. For $j < k$, we put

$$\Phi^i_{j,k} = \Phi^i_{k-1,k} \cdots \Phi^i_{j+1,j+2}\Phi^i_{j,j+1}$$

Then $\Phi^i_{j,k}$ is an epimorphism from $\mathrm{End}_R(e_{ij}R_R)$ to $\mathrm{End}_R(e_{ik}R_R)$.

**Proposition 3.1.** *$\Phi^i_{j,j+1}$ is an isomorphism if and only if $e_{ij}R_R$ is not a projective cover of $S(e_{i1}R_R)_R$.*

*Proof.* This is clear, since $\ker \Phi^i_{j,j+1} = \{\alpha \in \mathrm{End}_R(e_{ij}R_R) \mid \mathrm{Im}\ \alpha \subseteq S(e_{ij}R_R)\}$.

**Remark.** Henceforth we observe $R$ by representing it as

$$R = \begin{bmatrix} (e_{11}, e_{11}) & \cdots & (e_{mn(m)}, e_{11}) \\ (e_{11}, e_{12}) & \cdots & (e_{mn(m)}, e_{12}) \\ & \cdots & \\ (e_{11}, e_{mn(m)}) & \cdots & (e_{mn(m)}, e_{mn(m)}) \end{bmatrix} = \begin{bmatrix} e_{11}Re_{11} & \cdots & e_{11}Re_{mn(m)} \\ e_{12}Re_{11} & \cdots & e_{12}Re_{mn(m)} \\ & \cdots & \\ e_{mn(m)}Re_{11} & \cdots & e_{mn(m)}Re_{mn(m)} \end{bmatrix}$$

where $(e_{ij}, e_{kt}) = \mathrm{Hom}_R(e_{ij}R, e_{kt}R)$.

**Proposition 3.2.** *$R$ is left artinian.*

*Proof.* We may show that ${}_{eRe}eRf$ is artinian for all $e$, $f$ in $E$. We check this fact in four steps.

Step 1. As $R$ is semiprimary $QF$-3 and each $e_{i1}R_R$ is injective, we see from [1] that $e_{i1}Re_{j1}$ is artinian as a left $e_{i1}Re_{i1}$-module for all $i$, $j$.

Step 2. Assume that $e_{pq}R_R$ is a projective cover of $S(e_{i1}R_R)_R$. If $p \neq i$, Proposition 3.1 shows

$$e_{i1}Re_{i1} \overset{\Phi^i_{1,2}}{\cong} e_{i2}Re_{i2} \overset{\Phi^i_{2,3}}{\cong} \cdots \overset{\Phi^i_{n(i)-1,\,n(i)}}{\cong} e_{in(i)}Re_{in(i)}\ \text{(as ring)}$$

If $p = i$, Proposition 3.1 also shows

$$e_{i1}Re_{i1} \overset{\Phi^i_{1,2}}{\cong} e_{i2}Re_{i2} \overset{\Phi^i_{2,3}}{\cong} \cdots \overset{\Phi^i_{q-1,q}}{\cong} e_{iq}Re_{iq},$$

$$e_{i,q+1}Re_{i,q+1} \overset{\Phi^i_{q+1,q+2}}{\cong} \cdots \overset{\Phi^i_{n(i)-1,n(i)}}{\cong} e_{in(i)}Re_{in(i)},$$

$\Phi^i_{q,q+1}: e_{iq}Re_{iq} \to e_{i,q+1}Re_{i,q+1}$ is a ring epimorphism.

Since $e_{i1}Re_{i1}$ is artinian as a left $e_{i1}Re_{i1}$-module, $e_{i1}Re_{ij}$ is artinian as a left $e_{ij}Re_{ij}$-module for all $e_{ij}$.

Step 3. We observe $e_{ij}Re_{kt}$ for $i \neq k$. Put $f_j = e_{ij}$, $f_1 = e_{i1}$ ; $g_t = e_{kt}$, $g_1 = e_{k1}$. Then note that $f_jRg_t$ becomes a left $f_1Rf_1$-module by the epimorphism $\Phi^i_{1,j}: f_1Rf_1 \to f_jRf_j$. We define a mapping $\zeta$ from $f_jRg_t = (g_t, f_j)$ to $f_1Rg_t = (g_t, f_1)$ by the rule: $\alpha \to \theta^i_{j,1}\ \alpha$ for $\alpha \in (g_t, f_j)$. Then it is easy to check that $\zeta$ is a left $f_1Rf_1$-homomorphism. Moreover, noting $i \neq k$, we see that it is an isomorphism. On the other hand, the mapping $\eta: f_1Rg_1 = (g_1, f_1) \to f_1Rg_t = (g_t, f_1)$ given by the rule: $\alpha \to \alpha\theta^k_{t,1}$ for $\alpha \in (g_1, f_1)$ is a left $f_1Rf_1$-epimorphism. Hence $\zeta^{-1}\ \eta$ gives an epimorphism:

$$_{f_1Rf_1}f_1Rg_1 \to {}_{f_1Rf_1}f_jRg_t.$$

Since $_{f_1Rf_1}f_1Rg_1$ is artinian, it follows that $_{f_jRf_j}f_jRg_t$ is artinian.

Step 4. Put $e_1 = e_{i1}, \ldots, e_{n(i)} = e_{in(i)}$, and observe $e_jRe_k$ for $k \neq j$. If $k < j$, then $e_jRe_k$ becomes a left $e_kRe_k$-module by the epimorphism $\Phi^i_{k,j}$ : $e_kRe_k \to e_jRe_j$. Consider a mapping $\lambda$: $e_jRe_k = (e_k, e_j) \to e_kRe_k = (e_k, e_k)$ given by the rule: $\alpha \to \theta^i_{j,k}\alpha$ for $\alpha \in (e_k, e_j)$. As is easily seen, $\lambda$ is a left $e_kRe_k$-homomorphism and moreover it is a monomorphism. Since $_{e_kRe_k}e_kRe_k$ is artinian, it follows that $_{e_jRe_j}e_jRe_j$ is artinian. Next, if $k > j$, then $e_kRe_k$ becomes a left $e_jRe_j$-module by $\Phi^i_{j,k}$, and $_{e_jRe_j}e_jRe_k \simeq {}_{e_jRe_j}e_kRe_k$ by the mapping: $\alpha \to (\theta^i_{k,j})^{-1}\alpha$. Hence, in this case, we also see that $_{e_jRe_j}e_jRe_k$ is artinian, since $_{e_kRe_k}e_kRe_k$ is artinian.

By Steps 1–4, $eRf$ is artinian as a left $eRe$-module for all $e$, $f$ in $E$, so $R$ is left artinian.

**Lemma 3.3.** *Let $e$, $f$ be in $E$ such that $eR_R$ is injective and $fR_R$ is a projective cover of $S(eR_R)_R$. Put $X = Hom_R(fR, S(eR_R))$. Then*

1) *$_{eRe}X$ and $X_{fRf}$ are simple.*
2) *$_{eRe}X \simeq {}_{eRe}S(_{eRe}eRf)$ and $X_{fRf} \simeq S(eRf_{fRf})_{fRf}$.*
3) *$S(_{eRe}eRf) = S(eRf_{fRf})$.*

*Proof.* 1) It is enough to show that $eRe\alpha = \alpha fRf = X$ for any nonzero $\alpha$ in $X$. Let $0 \neq \beta \in X$. Since $fR_R$ is projective, there exists $\gamma$ in fRf satisfying $\alpha\gamma = \beta$: whence $X = \alpha fRf$. On the other hand, $\alpha$ and $\beta$ induce isomorphisms: $fR/\mathrm{J}(fR) \overset{\bar{\alpha}}{\simeq} \mathrm{S}(eR_R)$ and $fR/\mathrm{J}(fR) \overset{\bar{\beta}}{\simeq} \mathrm{S}(eR_R)$. Since any automorphism of $\mathrm{S}(eR_R)_R$ is induced from one of $eR_R$, we can obtain $\sigma$ in eRe such that $\bar{\beta}\bar{\alpha}^{-1} = \sigma$ on $\mathrm{S}(eR_R)$. Then $\bar{\beta} = \sigma\bar{\alpha}$ and it follows that $\beta = \sigma\alpha$. As a result, we see $X = eRe\alpha$. 2) and 3) are clear from 1).

**Lemma 3.4.** *Let $f$ and $g$ in $E$, and assume that $fR_R$ is a projective cover of $S(e_{i1}R_R)_R$ or, equivalently, $fR/J(fR) \simeq S(e_{i1}R)$.*

1) *For $\alpha$ in $(f, g)$ with $Im\ \alpha \supsetneqq S(gR)$, there exists $\beta_k$ in $(g, e_{ik})$ such that $Im\ \beta_k\alpha = S(e_{ik}R)$ for $k = 1,\dots, n(i)$.*

2) *If $g$ is in $E-\{e_{i1},\dots, e_{in(i)}\}$, then, for any $0 \neq \alpha \in (f, g)$, there exists $\beta_k \in (g, e_{ik})$ such that $Im\ \beta_k\alpha = S(e_{ik}R_R)$ for $k = 1,\dots, n(i)$.*

3) *If $g = e_{it}$, then, for any $0 \neq \alpha \in (f, g)$, there exists $\beta_k \in (g, e_{ik})$ such that $Im\ \beta_k\alpha = S(e_{ik}R_R)$ for $k = 1,\dots, t$.*

*Proof.* Note that 2) is contained in 1). For convenience's sake, put $e_1 = e_{i1},\dots, e_{n(i)} = e_{in(i)}$. Let $0 \neq \alpha \in (f, g)$. $\alpha$ induces an isomorphism $\bar{\alpha}$: $fR/\ker\alpha \simeq \mathrm{Im}\ \alpha$. Then, note $\ker\alpha \subseteq \mathrm{J}(fR)$. Since $fR/\mathrm{J}(fR) \simeq \mathrm{S}(e_kR)$, there exists an $R$-homomorphism $\gamma_k$ from $\mathrm{Im}\ \alpha$ onto $\mathrm{S}(e_kR)$. Put $\theta = \theta^i_{k,1}$; $e_kR \simeq \mathrm{J}_{k-1}(e_1R)$. Since $e_1R_R$ is injective, there exists $\delta_k \in (g, e_1)$ which is an extension of $\theta\gamma_k$, i.e., $\delta_k = \theta\gamma_k$ on $\mathrm{Im}\ \alpha$.

Now, if $\gamma_k$ is not a monomorphism, that is, $\mathrm{Im}\ \alpha \supsetneqq \mathrm{S}(gR)$, then $\mathrm{Im}\ \delta_k \subseteq \mathrm{J}_{n(i)}(e_1R) \subseteq \theta(e_kR)$; so $\theta^{-1}\delta_k \in (g, e_k)$ with $\mathrm{Im}\ \theta^{-1}\delta_k\alpha = \mathrm{S}(e_kR)$. Next, assume $g = e_t$ and $\mathrm{Im}\ \alpha = \mathrm{S}(e_tR)$. Then $\delta_k$ is a monomorphism with $\mathrm{Im}\ \delta_k = \mathrm{J}_{t-1}(e_1R) = \theta^i_{t,1}(e_tR)$. So, $\theta^{-1}\delta_k$ has a sense as a homomorphism from $e_tR_R$ to $e_kR_R$ for $k \leqq t$, and $\mathrm{Im}\ \theta^{-1}\delta_k\alpha = \mathrm{S}(e_kR)$. The proof is completed.

**Proposition 3.5.** *Let $f$ be in $E$, and assume that $fR_R$ is a projective cover of $S(e_{i1}R)_R$. Then*

1) *$S_k({}_RRf) = S(e_{i1}R_R)+\cdots+S(e_{ik}R_R)$ for $k = 1,\dots, n(i)$; whence $S_k({}_RRf)$ is a two sided ideal of $R$.*

2) *$S(e_{in(i)}R_R)(Rf/S_k({}_RRf)) = (S(e_{in(i)}R_R)+S_k({}_RRf))/S_k({}_RRf)$ for $k = 1,\dots, n(i)-1$.*

3) *$S(R_R)(Rf/S_{n(i)}({}_RRf)) = 0$*

*Therefore $Rf/S_k({}_RRf)$ is a non-small left $R$-module for $1 \leqq k < n(i)$ and $Rf/S_{n(i)}({}_RRf)$ is a small left $R$-module by Lemma 1.2.*

*Proof.* Put $e_k = e_{ik}$ for $k = 1, \dots, n(i)$. We observe $Rf$ and $e_k R$ by identifying these with $f$-column and $e_k$-row, respectively;

$$Rf = \begin{bmatrix} & \vdots & \\ & (f, e_1) & \\ 0 & \vdots & 0 \\ & (f, e_{n(i)}) & \\ & \vdots & \end{bmatrix}$$

$$e_k R = \begin{bmatrix} & & 0 & & \\ (e_{11}, e_k) & \cdots & (f, e_k) & \cdots & (e_{mn(m)}, e_k) \\ & & 0 & & \end{bmatrix}$$

We put $X_k = \mathrm{Hom}_R(fR, \mathrm{S}(e_k R))$. Then, by Lemma 3.4,

$$\mathrm{S}(e_k R) = \begin{bmatrix} & & & 0 & & & \\ 0 & \cdots & 0 & X_k & 0 & \cdots & 0 \\ & & & 0 & & & \end{bmatrix}$$

so we see

$$\mathrm{S}(e_1 R_R) + \cdots + \mathrm{S}(e_k R_R) = \begin{bmatrix} & 0 & \\ & \vdots & \\ & 0 & \\ & X_1 & \\ 0 & \vdots & 0 \\ & X_k & \\ & 0 & \\ & \vdots & \\ & 0 & \end{bmatrix}$$

On the other hand, using Lemma 3.4. we see

$$\mathrm{S}_k({}_R Rf) = \begin{bmatrix} & 0 & \\ & \vdots & \\ & 0 & \\ & X_1 & \\ 0 & \vdots & 0 \\ & X_k & \\ & 0 & \\ & \vdots & \\ & 0 & \end{bmatrix}$$

Hence $S(e_1R)_R+\cdots+S(e_kR_R) = S_k({}_RRf)$. The proofs of 2) and 3) are easy from 1).

**Proposition 3.6.** *The following conditions are equivalent for $f$ in $E$.*

1) *$fR_R$ is a projective cover of some $S(e_{i1}R_R)$.*

2) *${}_RRf$ is injective*

3) *${}_RRf$ is a non-small module*

*Proof.* 1) ⇒ 2). By Proposition 3.5 and Lemma 3.3, we see that $(e_{i1}R_R\,;\,{}_RRf)$ is an injective pair; whence ${}_RRf$ is injective (Lemma 1.1). 2) ⇒ 3) is clear. Assume that ${}_RRf$ is a non-small module. Then $S(R_R)Rf \neq 0$ by Lemma 1.2. Hence $(S(e_{i1}R)+\cdots+S(e_{in(i)}R))Rf \neq 0$ for some $i$. Let $g$ be in $E$ such that $gR_R$ is a projective cover of $S(e_{i1}R)$. By Proposition 3.5,

$$S(e_{i1}R)+\cdots+S(e_{in(i)}) = \begin{bmatrix} & 0 & \\ & \vdots & \\ & 0 & \\ & X_1 & \\ 0 & \vdots & 0 \\ & X_{n(i)} & \\ & 0 & \\ & \vdots & \\ & 0 & \end{bmatrix}$$

where $X_k = \operatorname{Hom}(gR, S(e_{ik}R))$, $k = 1,\dots, n(i)$. Hence if $g \neq f$, we see that $(S(e_{i1}R)+\cdots+S(e_{in(i)}R))Rf = 0$, a contradiction. Thus $f = g$ and hence $fR_R$ is a projective cover of $S(e_{i1}R)$.

Now we are in a position to show the following

**Theorem 3.7.** *$R$ is a left H-ring.*

*Proof.* By Proposition 3.2, $R$ is a left artinian ring. Let $f$ be in $E$, and assume that ${}_RRf$ is injective. Then, by Proposition 3.6, there exists $e_{i1}$ in E for which $fR_R$ is a projective cover of $S(e_{i1}R)_R$. By Proposition 3.5, $Rf/S_k(Rf)$ is a non-small left R-module for $k = 1,\dots, n(i)-1$, and $Rf/S_{n(i)}(Rf)$ is a small left $R$-module. Therefore, the proof is completed if we show that $Rf/S_k(Rf)$ is injective for $k=1,\dots, n(i)-1$. By Proposition 3.5, $S_k(Rf)$ is a two sided ideal of $R$. Here we denote the factor ring $R/S_k(Rf)$ by $\bar{R}$, and $r+S_k(Rf)$ by $\bar{r}$ for $r$ in $R$. We observe $\overline{Rf}$ by identifying it with

$$\begin{bmatrix} & \vdots & \\ & (f, e_{i1}) & \\ 0 & \vdots & 0 \\ & (f, e_{ik}) & \\ & (f, e_{i,k+1}) & \\ & \vdots & \end{bmatrix} \Bigg/ \begin{bmatrix} & 0 & \\ & \vdots & \\ & 0 & \\ & X_1 & \\ 0 & \vdots & 0 \\ & X_k & \\ & 0 & \\ & \vdots & \\ & 0 & \end{bmatrix}$$

where $X_j = \mathrm{Hom}_R(fR, \mathrm{S}(e_{ij}R))$, $j = 1,\dots, k$. As is easily seen, $(\bar{e}_{i,k+1}\bar{R}_{\bar{R}}, {}_{\bar{R}}\overline{Rf})$ is an injective pair; whence ${}_{\bar{R}}\overline{Rf}$ is injective (cf. Lemma 1.1). In order to show that $\overline{Rf}$ is injective as a left $R$-module, consider a diagram:

$$\begin{array}{c} 0 \to {}_RI \hookrightarrow {}_RR \\ \downarrow \phi \\ {}_R\overline{Rf} \end{array}$$

where ${}_RI \subseteq_e {}_RR$ and $\phi$ is an $R$-homomorphism. Put $\phi_1 = \phi$ and $I_1 = I$. It is easy to see that $\phi_1(\mathrm{S}_1(Rf)) = 0$ since

$$\mathrm{S}_1(Rf) = \begin{bmatrix} & 0 & \\ & \vdots & \\ & 0 & \\ 0 & X_1 & 0 \\ & 0 & \\ & \vdots & \\ & 0 & \end{bmatrix}$$

Since $\mathrm{S}_1(Rf) \subseteq I_1$ and $\phi_1(\mathrm{S}_1(Rf)) = 0$, $\phi_1$ induces an $R/\mathrm{S}_1(Rf)$-homomorphism $\phi_1^* : \mathrm{I}_1/\mathrm{S}_1(Rf) \to \overline{Rf}$. We can take a left ideal $I_2$ containing $I_1 + \mathrm{S}_2(Rf)$ and $R/\mathrm{S}_1(Rf)$-homomorphism $\phi_2 : I_2/\mathrm{S}_1(Rf) \to \overline{Rf}$ such that the restriction $\phi_2 \mid \mathrm{I}_1/\mathrm{S}_1(Rf)$ is $\phi_1^*$. Since $I_2 \supseteq \mathrm{S}_2(Rf)$ and $\phi_2(\mathrm{S}_2(Rf)) = 0$, $\phi_2$ induces an $R/\mathrm{S}_2(Rf)$-homomorphism $\phi_2^* : I_2/\mathrm{S}_2(Rf) \to \overline{Rf}$, and by the same argument, we can take a left ideal $I_3$ containing $I_2 + \mathrm{S}_3(Rf)$ and $R/\mathrm{S}_2(Rf)$-homomorphism $\phi_3 : I_3/\mathrm{S}_2(Rf) \to \overline{Rf}$ which is an extension of $\phi_2^*$. Continuing this procedure $k$ times, we obtain left ideals $I_1,\dots, I_k$ such that $\mathrm{S}_1(Rf) \subseteq \mathrm{I}_1$, $\mathrm{S}_2(Rf) + I_1 \subseteq \mathrm{I}_2,\dots, \mathrm{S}_k(Rf) + I_{k-1} \subseteq I_k$ and $R/\mathrm{S}_{i-1}(Rf)$-homomorphism $\phi_i : I_i/\mathrm{S}_{i-1}(Rf) \to \overline{Rf}$ such that $\phi_i \mid I_{i-1}/\mathrm{S}_{i-1}(Rf) = \phi_{i-1}^*$, where $\phi_{i-1}^*$ is the induced homomorphism from $\phi_{i-1}: I_{i-1}/S_{i-2}(Rf) \to \overline{Rf}$.

Here we consider the diagram:

$$\begin{array}{ccc} 0 \to I_k/S_{k-1}(Rf) & \hookrightarrow & \bar{R} \\ \downarrow \phi_k^* & & \\ \overline{Rf} & & \end{array}$$

Since ${}_{\bar{R}}\overline{Rf}$ is injective, there exists an $R$-homomorphism $\zeta: \bar{R} \to \overline{Rf}$ which is an extension of $\phi_k$. Let $\eta: R \to \bar{R}$ be the canonical homomorphism. Then $\zeta\eta$ is an extension of $\phi$. This completes the theorem.

**4. A remark on $(*)^*$.** Let $R$ be a basic semi-perfect ring with a complete set $E = \{e_{11}, \ldots, e_{1n(1)}, \ldots, e_{m1}, \ldots, e_{mn(m)}\}$ of orthogonal primitive idempotents satisfying

a) each $e_{i1}R_R$ is injective,

b) $e_{ij}R_R \simeq J_{j-1}(e_{i1}R_R)_R$ for $j = 1, \ldots, n(i)$ and $i = 1, \ldots, m$.

For this ring $R$, we show the following:

**Theorem 4.1.** *If $R$ is left or right artinian, then $R$ satisfies $(*)^*$, namely the following condition holds:*

c) *$J(e_{in(i)}R_R)_R$ is a singular module for $i = 1, \ldots, m$.*

*Proof.* Since $R$ is a perfect ring, a) and b) implies

d) All $S(e_{ij}R_R)_R$ are non-zero simple modules.

Since $R$ is a one sided artinian ring with a), b) and d), in view of arguments in § 3, we see that all results (except Proposition 3.2) in there are valid for this ring $R$. Now, in order to show c), we may show that $J(e_{in(i)}R_R)S(R_R) = 0$ for all $i$.

Let $f_i$ be in $E$ such that $f_iR_R$ is a projective cover of $S(e_{i1}R_R)_R$ for $i = 1, \ldots, m$. Put

$$X_k = \mathrm{Hom}_R(f_iR, S(e_{ik}R_R))$$

for $i = 1, \ldots, m$, $k = 1, \ldots, n(i)$. Then, as in § 3, we see

$$S(e_{ik}R)_R = \begin{bmatrix} & 0 & \\ & \vdots & \\ & 0 & \\ 0 & X_k & 0 \\ & 0 & \\ & \vdots & \\ & 0 & \end{bmatrix}$$

Noting this fact, we can easily calculate that $J(e_{in(i)}R_R)\,S(R_R) = 0$.

**5. Left H ⇒ right co-H.** In this section, we show that left $H$-rings are right co-$H$-rings. Since left $H$-rings and right co-$H$-rings are Morita invariant, we may show that basic left $H$-rings are right co-$H$. So, henceforth, we assume that $R$ is a basic left $H$-ring with $E$, a complete set of orthogonal primitive idempotents. Since $R$ is a semi-primary QF-3 (cf. Theorem 2.3), we can have a partition

$$E = \{e_{11}, \cdots, e_{1n(1)}\} \cup \cdots \cup \{e_{m1}, \ldots, e_{mn(m)}\} \cup \{g_1, \cdots, g_t\}$$

such that

a) each $e_{i1}R_R$ is injective,

b) $S(e_{i1}R_R)_R \simeq \cdots \simeq S(e_{in(i)}R_R)_R$ for all $i$,

c) $S(gR_R)$ is not simple for $g$ in $G = \{g_1, \ldots, g_t\}$.

As in § 3, we observe $R$ by identifying it with the matrix ring:

$$\begin{bmatrix} (e_{11}, e_{11}) & \cdots & (g_t, e_{11}) \\ & \cdots & \\ (e_{11}, g_t) & \cdots & (g_t, g_t) \end{bmatrix}$$

where $(p, q)$ means $\mathrm{Hom}_R(pR, qR)$ for $p, q$ in $E$.

**Lemma 5.1.** *Let $f$ be in $E$ and assume that $fR_R$ is a projective cover of $S(e_{i1}R_R)$. Then $(e_{i1}R_R\,;\,{}_RRf)$ is an injective pair, so ${}_RRf$ is injective.*

*Proof.* We put

$$A_1 = \begin{bmatrix} & 0 & 0 \\ & & \vdots & & \\ & & 0 & & \\ 0 & \cdots & 0 & \mathrm{Hom}_R(fR, S(e_{i1}R_R)) & 0 & \cdots & 0 \\ & & 0 & & \\ & 0 & \vdots & 0 & \\ & & 0 & & \end{bmatrix}$$

Then we see that $A_1$ is a two sided ideal of $R$, and furthermore, $A_1 = S(e_{i1}R_R)$ and $A_1 \subseteq S({}_RRf)$. Since $A_1Rf \neq \dot{0}$, ${}_RRf$ is non-small (cf. Lemma 1.2); whence ${}_RRf$ is injective and it follows that $A_1 = S({}_RRf)$. Since $A_1 = S(e_{i1}R_R) = S({}_RRf)$, clearly $(e_{i1}R_R\,;\,{}_RRf)$ is an injective pair.

**Lemma 5.2.** *Let $i$ be in $\{1,\dots,m\}$. If $n(i)=1$, then $S(e_{i1}R_R)_R \not\subseteq gR_R$ for all $g$ in $G$.*

*Proof.* Let $f$ be in $E$ such that $fR_R$ is a projective cover of $S(e_{i1}R_R)_R$. Assume that there exists $g$ in $G$ such that $S(e_{i1}R_R) \subseteq gR_R$. As in Lemma 5.1, we put

$$A_1=\begin{bmatrix} & & & 0 & & & \\ & & 0 & \vdots & & & \\ & & & 0 & & & \\ 0 & \cdots & 0 & \mathrm{Hom}_R(fR, S(e_{i1}R_R)) & 0 & \cdots & 0 \\ & & & 0 & & & \\ & & 0 & \vdots & & & \\ & & & 0 & & & \end{bmatrix}$$

$$A_g=\begin{bmatrix} & & & 0 & & & \\ & & 0 & \vdots & 0 & & \\ & & & 0 & & & \\ 0 & \cdots & 0 & \mathrm{Hom}_R(fR, S(gR_R)) & 0 & \cdots & 0 \\ & & & 0 & & & \\ & & 0 & \vdots & 0 & & \\ & & & 0 & & & \end{bmatrix}$$

Then, as we saw in Lemma 5.1, $A_1=S(e_{i1}R_R)=S({}_RRf)$ and note that $A_g$ is a right ideal of $R$ with $A_g \subseteq S(gR_R)$. Since $A_g(Rf/S(Rf)) \neq 0$, $Rf/S(Rf)$ is non-small; whence $Rf/S(Rf)$ is injective as a left $R$-module. Consider the factor ring $\bar{R}=R/A_1(=R/S(Rf))$. Since ${}_R\overline{Rf}$ is injective, ${}_{\bar{R}}\overline{Rf}$ is also injective. Hence there must exist $h$ in $E$ such that $(\bar{h}\bar{R}_{\bar{R}}\,;\,{}_{\bar{R}}\overline{Rf})$ becomes an injective pair. As is easily seen, $h \neq e_{i1}$. Further, we see from $n(i)=1$ that $h \notin E-(G\cup\{e_{i1}\})$; whence $h$ must be in $G$. Since $\bar{h}\bar{R}_{\bar{R}}$ is injective, $S(\bar{h}\bar{R}_{\bar{R}})$ has the simple socle. However, this shows that $S(hR_R)_R$ has the simple socle, a contradiction. Thus $S(e_{i1}R)_R \not\subseteq gR$ for all $g$ in $G$.

**Lemma 5.3.** *Let $i\in\{1,\dots,m\}$ and assume that $n(i)\geq 2$ and $e_{i2}R_R \not\subseteq e_{ij}R_R$ for $j=3,\dots,n(i)$. Then*

1) *$e_{i2}R_R \not\subseteq gR_R$ for all $g$ in $G$,*

2) *$e_{i2}R_R \simeq J(e_{i1}R_R)_R$.*

*Proof.* We take $f$ in $E$ such that $fR_R$ is the projective cover of $S(e_{i1}R_R)_R$. We put

$$A_1 = \begin{bmatrix} & & & 0 & & & \\ & 0 & & \vdots & & 0 & \\ & & & 0 & & & \\ 0 & \cdots & 0 & \mathrm{Hom}_R(fR,\ \mathrm{S}(e_{i1}R_R)) & 0 & \cdots & 0 \\ & & & 0 & & & \\ & 0 & & \vdots & & 0 & \\ & & & 0 & & & \end{bmatrix}$$

$$A_2 = \begin{bmatrix} & & & 0 & & & \\ & 0 & & \vdots & & 0 & \\ & & & 0 & & & \\ 0 & \cdots & 0 & \mathrm{Hom}_R(fR,\ \mathrm{S}(e_{i2}R_R)) & 0 & \cdots & 0 \\ & & & 0 & & & \\ & 0 & & \vdots & & 0 & \\ & & & 0 & & & \end{bmatrix}$$

Then $A_1 = \mathrm{S}(e_{11}R_R) = \mathrm{S}({}_RRf)$ by the proof of Lemma 5.1 and $A_2$ is a right ideal of $R$ with $A_2 \subseteq \mathrm{S}(e_{12}R_R)$. Put $\bar{R} = R/A_1$. Then note that $\mathrm{J}(e_{i1}R_R)$ and $e_{i2}R$ become canonically right $\bar{R}$-module, and there exists a canonical isomorphism: $\bar{e}_{i2}\bar{R}_{\bar{R}} \simeq e_{i2}R_{\bar{R}}$.

Since $A_2\overline{Rf} \neq 0$, ${}_RRf$ is non-small; so $\overline{Rf}$ is injective as a left $R$-module and hence so is as a left $\bar{R}$-module. Thus there exists $h$ in $E$ for which $(\bar{h}\bar{R}_{\bar{R}}\,;\ {}_{\bar{R}}\overline{Rf})$ is an injective pair. Then, as is easily seen, $h \neq e_{i1}$ and $h \neq e_{kt}$ for any $kt$ with $k \neq i$. Thus $h = e_{i2}$ or $h \in G$.

Here, assume that there exists $g$ in G such that $e_{i2}R_R \subsetneq gR_R$. Then we see that $h \neq e_{i2}$; whence $h \in G$. However, as in the proof of Lemma 5.2, this implies that $hR_R$ has the simple socle, a contradiction. Thus $e_{i2}R_R \not\subsetneq gR_R$ for all $g$ in $G$.

Now, if $h$ is in $G$, then $hR_R$ has the simple socle as above, a contradiction. As a result, $h$ must be $e_{i2}$, whence $\bar{e}_{i2}\bar{R}_{\bar{R}}$ is injective. Since $\bar{e}_{i2}\bar{R}_{\bar{R}} \simeq e_{i2}R_{\bar{R}} \subseteq_e \mathrm{J}(e_{i1}R_R)_{\bar{R}}$, it follows that $e_{i2}R_{\bar{R}} \simeq \mathrm{J}(e_{i1}R_R)_{\bar{R}}$; whence $e_{i2}R_R \simeq \mathrm{J}(e_{i1}R_R)_R$ as desired.

**Lemma 5.4.** 1) *There is a permutation* $\{e_{\rho(i2)},\dots,e_{\rho(in(i))}\}$ *of* $\{e_{i2},\dots,e_{in(i)}\}$ *such that*

$$J_{k-1}(e_{i1}R_R)_R \simeq e_{\rho(ik)}R_R$$

*for* $k = 2,\dots,n(i)$.

2) $e_{ij}R_R \not\subseteq gR_R$ *for all* $j$ *and* $g$ *in* $G$.

3) $S(e_{i1}R_R)_R \not\subseteq gR_R$ *for all* $g$ *in* $G$; *whence* $G$ *must be empty.*

*Proof.* 1) and 2) are shown by the same proof as in the proof of Lemma 5.3. In order to prove 3), we take $f$ in $E$ such that $fR_R$ is a projective cover of $S(e_{i1}R_R)_R$. By 1) and 2) (cf. Proofs of Lemma 5.1 and 5.3), we see that

$$S_{n(i)}({}_R Rf) = S(e_{i1}R_R) + \cdots + S(e_{in(i)}R_R)$$

Now, assume that there exists $g$ in G such that $S(e_{i1}R_R) \subseteq S(gR_R)_R$. Then $S(gR_R)(Rf/S_{n(i)}({}_R Rf)) \neq 0$; whence $Rf/S_{n(i)}({}_R Rf)$ is non-small and it is injective as a left $R$-module. Put $\bar{R} = R/S_{n(i)}({}_R Rf)$. Then $\overline{Rf}$ is injective as a left $\bar{R}$-module. So, there exists $h$ in $E$ such that $(\bar{h}\bar{R}_{\bar{R}}\,;\, {}_{\bar{R}}\overline{Rf})$ is an injective pair. Then, as is easily seen, $h$ is in $G$. Since $\bar{h}\bar{R}_{\bar{R}}$ has the simple socle, $hR_R$ has also the simple socle, a contradiction. Thus 3) holds.

We are now ready to show the following

**Theorem 5.5.** *R is a right co-H-ring.*

*Proof.* By Theorem 2.4, $R$ satisfies the *ACC* on right annihilator ideals. And, by Theorems 2.2 and 4.1 and Lemma 5.4, we see that $R$ satisfies $(*)^*$. So, $R$ is a right co-H-ring.

**6. An application of H-rings.** In this section, we show the following

**Theorem 6.1.** *If R is a right QF-*3 *and right generalized uniserial ring then R is a generalized uniserial ring.*

**Remark.** A ring $R$ is said to be right (left) generalized uniserial if it is right (left) artinian and, for any primitive idempotent $e$, $eR_R$ (${}_R Re$) has a unique composition series. A left and right generalized uniserial ring is said to be simply generalized uniserial.

For a proof of Theorem 6.1, the following two lemmas are needed.

**Lemma 6.2.** *If R is a right QF-*3 *and right generalized uniserial ring, then R is a right co-H- (hence left H-) ring.*

*Proof.* This is easily shown by Theorem 2.2.

**Lemma 6.3.** *If R is a right QF-*3 *and right generalized uniserial* *ring, then so is the factor ring $R/S(R_R)$.*

*Proof.* We can assume that $R$ is a basic ring. By Lemma 6.1, $R$ is a right co-$H$-ring; so, as in § 3, we observe $R$ by identifying it with the matrix ring:

$$\begin{bmatrix} (e_{11}, e_{11}) & \cdots & (e_{mn(m)}, e_{11}) \\ & \cdots & \\ (e_{11}, e_{mn(m)}) & \cdots & (e_{mn(m)}, e_{mn(m)}) \end{bmatrix}$$

where $E = \{e_{11}, \dots, e_{1n(1)}, \dots, e_{m1}, \dots, e_{mn(m)}\}$ is a complete set of orthogonal primitive idempotents such that

a) each $e_{i1}R_R$ is injective,

b) $e_{in(i)}R_R \subsetneq e_{i,n(i)-1}R_R \subsetneq \cdots \subsetneq e_{i2}R_R \subsetneq e_{i1}R_R$

for i $= 1, \dots, m$.

We put $\overline{R} = R/S(R_R)$, and for $r$ in $R$, $\bar{r} = r + S(R_R)$. Then $\overline{R}$ is also a basic and $\{\bar{e}_{ij} \mid \bar{e}_{ij} \neq 0\}$ is a complete set of orthogonal primitive idempotents. We see that

c) $\bar{e}_{in(i)}\overline{R}_{\overline{R}} \subsetneq \bar{e}_{i,n(i)-1}\overline{R}_{\overline{R}} \subsetneq \cdots \subsetneq \bar{e}_{i2}\overline{R}_{\overline{R}}$

for $i = 1, \dots, m$.

When $\bar{e}_{i1}\overline{R} \neq 0$, we take $f$ in $E$ such that $fR_R$ is the projective vover of $S_2(e_{i1}R_R)_R$. Then we can take $(\bar{e}_{i1}\overline{R}_{\overline{R}} ; {}_{\overline{R}}\overline{Rf})$ is an injective pair. As a result

d) each $\bar{e}_{i1}\overline{R}_{\overline{R}}$ is injective if it is non-zero.

By c) and d), we see that $\overline{R}$ is a right co-H-ring; so $\overline{R}$ is $QF$-3 (cf. Theorems 2.2 and 4.1). As $R$ is clearly right generalized uniserial, this completes the proof.

*Proof of Theorem* 6.1 : We can assume that $R$ is a basic ring. Let $E = \{e_{ij}\}$ be as in Lemma 6.3. We take $f_i$ in E such that $f_iR_R$ is the projective cover of $S(e_{i1}R_R)_R$ for $i = 1, \dots, m$. Then, by Proposition 3.5,

1) $S(e_{i1}R_R) + \cdots + S(e_{ik}R_R) = S_k({}_RRf_i)$

for $k = 1, \dots, n(i)$ and moreover we see

2) $S_k({}_RRf_i)/S_{k-1}({}_RRf_i)$ is simple

as a left $R$-module for $k = 1, \dots, n(i)$.

Now, by the induction of the sum of composition lengths of all $e_{ij}R_R$ together with Lemma 6.3, $R/S(R_R)$ is a generalized uniserial ring. Here, in view of 1) and 2) above, this implies that $R$ is left uniserial.

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