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SEPARATED SETS OF TORSION THEORIES

Dedicated to Prof. Hiroyuki Tachikawa on the occasion of his 60th birthday

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Throughout the following R will denote an associative ring with identity element 1 and R-mod will denote the category of unitary left R-modules. The frame of all (hereditary) torsion theories on R-mod will be denoted by R-tors. Notation and terminology concerning R-tors will follow [2]. In particular, if M is a left R-module then $\mathrm{E}(M)$ will denote the injective hull of M, $\xi(M)$ will denote the smallest torsion theory on R-mod relative to which M is torsion and $\chi(M)$ will denote the largest torsion theory on R-mod relative to which M is torsionfree. If $\tau \in R$-tors then a nonzero left R-module N is *τ-cocritical* if N is τ-torsionfree but every proper homomorphic image of N is τ-torsion. A τ-cocritical left R-module N is uniform and has the property, which we will use repeatedly, that $\chi(N') = \chi(N)$ for every nonzero submodule N' of N. If $\sigma \leqq \tau$ in R-tors we say that τ is a *generalization* of σ. The generalization is *proper* if $\sigma < \tau$.

The notion of a separated set of torsion theories on R-mod was considered briefly in Chapter 29 of [2]. We expand the definition given there as follows: if σ is a torsion theory on R-mod then a set U of generalizations of σ in R-tors is *σ-separated* if and only if $\tau \wedge [\vee(U\setminus\{\tau\})] = \sigma$ for each τ in U. The empty set is trivially σ-separated for every torsion theory σ. This relation was studied in the context of modular lattices (under the name of "independence") in [3] and [4]. It is straightforward to show that the following result is true:

1. Proposition. *If $\sigma \in R$-tors then the following conditions on a nonempty set U of generalizations of σ are equivalent:*

(1) *U is σ-separated;*

(2) *For any partition $U = U' \cup U''$ of U, $\sigma = (\vee U') \wedge (\vee U'')$;*

(3) *Every nonempty subset of U is σ-separated;*

(4) *Every finite nonempty subset of U is σ-separated.*

The torsion theory σ has *finite dimension* if and only if every σ-separated set of generalizations of σ is finite.

2. **Proposition.** *If* $\sigma \in R$*-tors and if* $\{U_i \mid i \in \Omega\}$ *is a chain of* σ*-separated sets* of *generalizations of* σ *then* $U = \cup\{U_i \mid i \in \Omega\}$ *is* σ*-separated.*

Proof. If Y is a finite subset of U then there exists some $h \in \Omega$ such that $Y \subseteq U_h$ and so Y is σ-separated. Hence, by Proposition 1, U is σ-separated. □

By Zorn's Lemma, we see that if $\sigma \in R$-tors then any σ-separated set of generalizations of σ is contained in a maximal σ-separated set.

For the purposes of this note we will say that a torsion theory σ on R-mod is *good if* and only if for each torsion theory τ on R-mod satisfying $\sigma < \tau$ there exists a σ-cocritical τ-torsion left R-module. By Proposition 2.10 of [2] we know that ξ, the unique minimal element of R-tors, is always good.

Recall that if $\sigma \in R$-tors then the ring R is *σ-noetherian* if and only if the set of all σ-pure left ideals of R satisfies the ascending chain condition. If σ is a torsion theory on R-mod such that R is σ-noetherian then we claim that σ is good. Indeed, assume that $\sigma < \tau$ in R-tors and let N be a nonzero τ-torsion σ-torsionfree left R-module. If $0 \neq x \in N$ then the set of all σ-pure left ideals of R containing $(0 : x)$ is nonempty and so has a maximal element H. Then R/H is τ-torsion and σ-cocritical, proving that σ is good.

Thus we see, in particular, that if the ring R is left noetherian then every torsion theory on R-mod is good.

If $\sigma < \tau$ in R-tors then we say that the torsion theory τ is *σ-uniform* if and only if the set of torsion theories τ' satisfying $\sigma < \tau' \leqq \tau$ is closed under taking finite meets. We will denote the set of all σ-uniform torsion theories on R-mod by σ-unif.

3. **Proposition.** *Let* σ *be a good torsion theory on R-mod and let* τ *be a proper generalization of* σ *in R-tors. Then the following conditions are equivalent:*

(1) *τ is σ-uniform;*

(2) *$\chi(M) = \chi(M')$ for all τ-torsion σ-cocritical left R-modules M and M'.*

Proof. (1) ⇒ (2) : If M and M' are τ-torsion σ-cocritical left R-modules we set $\rho = \sigma \vee \xi(M)$ and $\rho' = \sigma \vee \xi(M')$. Then $\sigma \neq \rho$, $\rho' \leqq \tau$ and so, by (1), $\sigma \neq \rho \wedge \rho' = \sigma \vee [\xi(M) \wedge \xi(M')]$. Since σ is good, there exists a σ-cocritical left R-module N which is $(\rho \wedge \rho')$-torsion, and so $\sigma \vee \xi(N) \leqq \rho \wedge \rho'$. In particular, N is not $[\xi(M) \wedge \xi(M')]$-torsionfree and so, by restriction if necessary, we can assume that it is $[\xi(M) \wedge \xi(M')]$-torsion. This means that there exist nonzero R-homomorphisms from M to $E(N)$ and from M' to $E(N)$ which must indeed be monic since N is σ-torsionfree and M, M' are σ-cocritical. Since N is cocritical and hence uniform as well, this implies that $\chi(M) = \chi(M')$.

(2) ⇒ (1) : Assume that $\sigma < \tau'$, $\tau'' \leqq \tau$. Since σ is good we know that there exist σ-cocritical left R-modules M' and M'' satisfying $\xi(M') \leqq \tau'$ and $\xi(M'') \leqq \tau''$. By (2), this means that $\chi(M') = \chi(M'')$ and so we can assume that $E(M') = E(M'')$. If $N = M' \cap M''$ then N is σ-cocritical and $\sigma \neq \sigma \vee \xi(N) \leqq \tau' \wedge \tau''$, proving (1). □

If $\sigma \in R$-tors, let us denote by $\mathbf{M}(\sigma)$ the set of all prime torsion theories of R-mod of the form $\chi(M)$, where M is a σ-cocritical left R-module. By Proposition 33.21 of [2] we see that $\mathbf{M}(\sigma)$ is contained in the set $\mathbf{P}_0(\sigma)$ of all minimal prime generalizations of σ. However, we need not have equality in general. If $\sigma \leqq \tau$ in R-tors, let $\mathbf{M}(\sigma, \tau)$ be the set of all elements of $\mathbf{M}(\sigma)$ of the form $\chi(M)$, where M is both σ-cocritical and τ-torsion. The cardinality of $\mathbf{M}(\sigma, \tau)$ will be called the *σ-rank* of τ. Clearly $\mathbf{M}(\sigma, \sigma) = \phi$ and, by definition, we see that the set $\mathbf{M}(\sigma, \tau)$ is nonempty (and hence the σ-rank of τ is nonzero) if $\sigma < \tau$ and σ is good. We also note that if $\sigma \leqq \tau' \leqq \tau$ then $\mathbf{M}(\sigma, \tau') \subseteq \mathbf{M}(\sigma, \tau)$. By Proposition 3, we see that if $\sigma < \tau$ and σ is good then τ is σ-uniform if and only if it has σ-rank equal to 1, i.e. if and only if $\mathbf{M}(\sigma, \tau)$ is a singleton $\{\pi\}$. In this case, we will say that the σ-uniform torsion theory τ is of *type* π.

4. **Proposition.** *Let σ be a good torsion theory on R-mod. A set U of σ-uniform torsion theories is σ-separated if and only if no two elements of U are of the same type.*

Proof. Assume that no two elements of U are of the same type. By Proposition 1 it suffices to assume that the set U is nonempty and finite. Let $U = \{\tau_1, \cdots, \tau_n\}$ be a set of σ-uniform torsion theories and let $\mathbf{M}(\sigma, \tau_i) = \{\pi_i\}$ for each $1 \leqq i \leqq n$. For each such i, let M_i be a τ_i-torsion σ-cocritical left R-module satisfying $\chi(M_i) = \pi_i$. If there exists an index h such that $\tau_h \wedge [\vee_{i \neq h} \tau_i] \neq \sigma$ then there exists a σ-cocritical left R-module N which is $\tau_h \wedge [\vee_{i \neq h} \tau_i]$-torsion. In particular, N is τ_h-torsion and so $\chi(N) \in \mathbf{M}(\sigma, \tau_h) = \{\pi_h\}$. Hence, without loss of generality, we can assume that $N = M_h$. Since M_h is $[\vee_{i \neq h} \tau_i]$-torsion, there exists an index $k \neq h$ such that M_h is not τ_k-torsionfree. Replacing M_h by its τ_k-torsion submodule, we can assume that it is τ_k-torsion and so $\pi_h = \chi(M_h) \in \mathbf{M}(\sigma, \tau_k) = \{\pi_k\}$, contradicting the assumption that τ_h and τ_k are not of the same type.

Now, conversely, assume that U is σ-separated. If there are two distinct elements τ and τ' of U of the same type $\chi(M)$, then $\xi(M) \leq \tau \wedge [\vee(U \setminus \{\tau\})]$, contradicting σ-separation. Therefore no two elements of U are of the same type. □

5. Corollary. *If σ is a good torsion theory on R-mod then any two maximal σ-separated sets of σ-uniform torsion theories have the same cardinality*

Proof. The cardinality of a maximal σ-separated set of σ-uniform torsion theories is clearly equal to the cardinality of $\mathbf{M}(\sigma)$. □

Recall that an *independence structure* $\mathscr{E}$ on a nonempty set A consists of a family of subsets of A satisfying the following conditions :

(1) $\phi \in \mathscr{E}$;

(2) If $A' \subseteq A'' \in \mathscr{E}$ then $A' \in \mathscr{E}$;

(3) If A' and A'' are finite sets in $\mathscr{E}$ satisfying $|A'| < |A''|$ then there is a set B in $\mathscr{E}$ satisfying $A' \subseteq B \subseteq A' \cup A''$ and $|B| = |A''|$;

(4) If every finite subset A' of a set A belongs to $\mathscr{E}$ then $A \in \mathscr{E}$.

For more information on such structures, refer to [1] or [5].

6. Proposition. *If σ is a good torsion theory on R-mod then the family of all σ-separated sets of σ-uniform torsion theories is an independence structure on σ-unif.*

Proof. Condition (1) is true by definition and conditions (2) and (4) follow from Proposition 1. We are therefore left to prove condition (3). Let U' and U'' be finite σ-separated sets of σ-uniform torsion theories on R-mod with $|U'| < |U''|$. Say $U' = \{\tau_1, \cdots, \tau_k\}$, where each τ_i is of type π_i, and let $U'' = \{\sigma_1, \cdots, \sigma_n\}$, where each σ_j is of type π'_j. By renumbering if necessary, we can assume that there exists an integer $1 \leq t \leq k+1$ such that π_i and π'_i are equal for all $i < t$ and are not equal for all $i \geq t$. Then $Y = \{\tau_1, \cdots, \tau_k, \sigma_{k+1}, \cdots, \sigma_n\}$ is a set of σ-uniform torsion theories no two of which are of the same type and so the set is σ-separated. Moreover, $U' \subseteq Y \subseteq U' \cup U''$ and $|Y| = |U''|$. □

If $\sigma < \sigma' \leqq \tau$ in R-tors then τ is *σ-essential* over σ' if and only if $\sigma \neq \sigma' \wedge \sigma''$ for all $\sigma < \sigma'' \leqq \tau$. Thus, trivially, if $\sigma < \tau$ then τ is σ-essential over itself. Moreover, a torsion theory $\tau > \sigma$ is σ-uniform if and only if it is σ-essential over every torsion theory σ' satisfying $\sigma < \sigma' \leqq \tau$.

7. Proposition. *If $\sigma < \sigma' \leqq \tau$ are torsion theories on R-mod with σ good then τ is σ-essential over σ' if and only if* $\mathbf{M}(\sigma, \tau) = \mathbf{M}(\sigma, \sigma')$.

Proof. Assume that τ is σ-essential over σ'. Since $\sigma' \leqq \tau$ we have $\mathbf{M}(\sigma, \sigma') \subseteq \mathbf{M}(\sigma, \tau)$. On the other hand, assume that $\pi \in \mathbf{M}(\sigma, \tau)$ and let N be a σ-cocritical τ-torsion left R-module satisfying $\pi = \chi(N)$. Then $\sigma < \sigma \vee \xi(N) \leqq \tau$ so $\sigma \neq \sigma' \wedge (\sigma \vee \xi(N))$. Since σ is good, there exists a σ-cocritical left R-module N' which is $[\sigma' \wedge (\sigma \vee \xi(N))]$-torsion. In particular, there exists a nonzero R-homomorphism from N to $E(N')$, which must be monic since N and N' are σ-cocritical. Then the uniformness of N' implies that $\pi = \chi(N) = \chi(N') \in \mathbf{M}(\sigma, \sigma')$, proving that $\mathbf{M}(\sigma, \tau) = \mathbf{M}(\sigma, \sigma')$.

Conversely, assume that $\mathbf{M}(\sigma, \tau) = \mathbf{M}(\sigma, \sigma')$ and let $\sigma < \sigma'' \leqq \tau$. If N is a σ-cocritical σ''-torsion left R-module then $\chi(N) \in \mathbf{M}(\sigma, \tau) = \mathbf{M}(\sigma, \sigma')$ and so N is σ'-torsion as well. Thus $\sigma < \sigma \vee \xi(N) \leqq \sigma' \wedge \sigma''$, proving that τ is σ-essential over σ'. □

8. Corollary. *If $\sigma \in$ R-tors is good then a σ-essential generalization of a σ-uniform torsion theory is σ-uniform.*

Proof. This is a direct consequence of Proposition 3 and Proposition 7. □

9. Corollary. *If $\sigma \in$ R-tors is good and if $\tau > \sigma$ in R-tors then there exists a σ-separated set U of σ-uniform torsion theories on R-mod such that τ is a σ-essential extension of $\vee U$.*

Proof. Take $U = \{\sigma \vee \xi(M) \mid M$ a σ-cocritical τ-torsion left R-module$\}$. This set is σ-separated by Proposition 4. □

10. Proposition. *If σ is a good torsion theory on R-mod then any proper generalization of σ has a unique maximal σ-essential generalization.*

Proof. Let $\sigma < \tau$ in R-tors and let $\tau' = \vee\{\sigma' > \sigma \mid \mathbf{M}(\sigma, \tau) = \mathbf{M}(\sigma, \sigma')\}$. Clearly $\tau' \geqq \tau$ and so $\mathbf{M}(\sigma, \tau) \subseteq \mathbf{M}(\sigma, \tau')$. Conversely, if

$\pi \in \mathbf{M}(\sigma, \tau')$ and if M is a σ-cocritical τ'-torsion left R-module satisfying $\pi = \chi(M)$ then there exists a torsion theory $\sigma' > \sigma$ satisfying $\mathbf{M}(\sigma, \tau) = \mathbf{M}(\sigma, \sigma')$ and having the property that M is not σ'-torsionfree. Therefore $\pi = \chi(M')$, where M' is the σ'-torsion submodule of M. This implies that $\pi \in \mathbf{M}(\sigma, \sigma') = \mathbf{M}(\sigma, \tau)$, proving that $\mathbf{M}(\sigma, \tau')$ and $\mathbf{M}(\sigma, \tau)$ are equal. By Proposition 7, this means that τ' is a σ-essential generalization of τ which, by construction, is clearly maximal. □

If $\sigma \leqq \tau$ in R-tors we will denote the σ-rank of τ by $r_\sigma(\tau)$. The torsion theory τ is *σ-flat* if and only if $r_\sigma(\tau') > r_\sigma(\tau)$ for all $\tau' > \tau$ in R-tors. In other words, τ is σ-flat if and only if for each $\tau' > \tau$ there exists a τ'-torsion σ-cocritical left R-module which is τ-torsionfree. (Note that the term "flat" is used here in its combinatoric, rather than algebraic, sense ; see [1] or [5].)

11. Proposition. *If $\sigma \leqq \tau$ in R-tors then the family of all σ-flat generalizations τ is closed under taking arbitrary meets.*

Proof. Let U be a nonempty set of σ-flat generalizations of τ and assume that $\tau' > \wedge U$ in R-tors. Then there exists an element ρ of U such that $\tau' \nleqq \rho$ and so $\tau' \vee \rho > \rho$. Thus there exists a ρ-torsionfree σ-cocritical left R-module M which is $(\tau' \vee \rho)$-torsion and hence not τ'-torsionfree. Replacing M by its τ'-torsion submodule, we can assume that it is τ'-torsion. On the other hand, M is $(\wedge U)$-torsionfree since it is ρ-torsionfree. Therefore $r_\sigma(\tau') > r_\sigma(\wedge U)$. □

12. Proposition. *Let σ be a good torsion theory on R-mod and let $\sigma < \tau$. If $\mathbf{M}(\sigma, \tau)$ is finite then the maximal σ-essential generalization of τ is the meet of all σ-flat generalizations of τ.*

Proof. Let τ' be the maximal σ-essential generalization of τ. If $\tau'' > \tau'$ then τ'' is not σ-essential over τ and so, by Proposition 7, $\mathbf{M}(\sigma, \tau'') \supset \mathbf{M}(\sigma, \tau) = \mathbf{M}(\sigma, \tau')$. Therefore $r_\sigma(\tau'') > r_\sigma(\tau')$, proving that τ' is σ-flat over τ. Now assume that ρ is σ-flat over τ and satisfies $\rho < \tau'$. Then $r_\sigma(\tau') > r_\sigma(\rho)$ and so $\mathbf{M}(\sigma, \tau) = \mathbf{M}(\sigma, \tau') \supset \mathbf{M}(\sigma, \rho) \supseteq \mathbf{M}(\sigma, \tau)$, which is a contradiction. Thus, by Proposition 11, τ' is the meet of all σ-flat generalizations of τ. □

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