Math. J. Okayama Univ. 31 (1989), 141−151

ON HEREDITARY ARTINIAN RINGS AND AZUMAYA'S EXACTNESS CONDITION

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Azumaya [3] and Morita [9] established the theory of Morita duality and proved that there is a duality between the categories of finitely generated left R-modules over a ring R and finitely generated right S-modules over some ring S if and only if R is left artinian and the indecomposable injective left R-modules are all finitely generated. An interesting problem is to determine the type of artinian rings which posses self-duality, i.e., a duality between their categories of finitely generated left and right modules. An artinian ring has self-duality when its basic ring is isomorphic to the endomorphism ring of its minimal cogenerator.

In this paper, we characterize left artinian hereditary rings via module diagrams, and employ this and a result of Habeb [8] to show that a basic exact hereditary ring and the endomorphism ring of its minimal cogenerator have the same algebra diagrams, thus lending support to Azumaya's conjecture that his exact rings have self-duality.

Azumaya [4] called a left artinian ring R exact in case it has a composition series of two-sided ideals

$$ {}_RR_R = I_0 > I_1 > \cdots > I_n = 0 $$

such that for all i every left endomorphism of I_{i-1}/I_i is given by right multiplication by an element of R. He proved that this notion is left-right symmetric and that Nakayama's serial (or generalized uniserial) rings are exact. Extending this notion to any bimodules, Camillo, Fuller and Haack [6] said that a bimodule that has a composition series whose composition factors are balanced is an *exact module*, so that a ring R is an *exact ring* in case the regular bimodule ${}_RR_R$ is exact.

A *module diagram* $\mathcal{M}$ is a finite directed graph with distinguished node 0 such that : (i) there is at most one arrow between any two nodes ; (ii) there are no oriented cycles in $\mathcal{M}$ and no arrows emanating from 0 ; (iii) if $x \neq 0$ then $x \to 0$ in $\mathcal{M}$ if and only if there is no arrow $x \to y \neq 0$ in $\mathcal{M}$. (Cf. [1], [7].) We also let $\mathcal{M}$ denote the set of nodes in the diagram $\mathcal{M}$. A subdiagram $\mathcal{U} \leq \mathcal{M}$ satisfies $x \in \mathcal{U}$ and $x \to y$ implies $y \in \mathcal{U}$, and for any subset $X \subseteq \mathcal{M}$, $\mathcal{U}(X)$ denotes the smallest subdiagram of $\mathcal{M}$ containing X. The *radical* of $\mathcal{M}$, Rad($\mathcal{M}$), is the intersection of all maximal subdiagrams of $\mathcal{M}$. A diagram $\mathcal{M}$ is called *tree* in case for each $x \in \mathcal{M}$ no two distinct paths in $\mathcal{U}(x)$ end at the same node $z \neq 0$.

A module diagram $\mathcal{M}$ is called an n-diagram if there is a function

$$\lambda : \mathcal{M}\backslash\{0\} \to \{1,\dots, n\}$$

labeling the nodes of $\mathcal{M}$. If $\mathcal{M}$ and $\mathcal{N}$ are both n-diagrams, a homomorphism $\phi: \mathcal{M} \to \mathcal{N}$ is a function on nodes satisfying (i) if $\phi(x) \to v$ in $\mathcal{N}$ then there is a $y \in \mathcal{M}$ with $x \to y$ and $\phi(y) = v$; (ii) if $x \to y$ in $\mathcal{M}$ and $\phi(y) \neq 0$ then $\phi(x) \to \phi(y)$ in $\mathcal{N}$; and (iii) $\lambda(\phi(x)) = \lambda(x)$ whenever $\phi(x) \neq 0$.

Throughout this paper R is a left artinian ring with radical J, basic set of primitive idempotents $e_1,\dots, e_n$, and simple left modules $S_i = Re_i/Je_i$, $i = 1,\dots, n$.

As in [1], [7], a pair $(\mathcal{M}, \delta)$ is a diagram for a finitely generated left R-module M in case : $\mathrm{Card}(\mathcal{M}\backslash\{0\}) = c(M)$, the composition length of M; $\mathcal{M}$ is an n-diagram with label λ; and $\delta: \mathcal{L}(\mathcal{M}) \to \mathcal{L}(M)$ is an injective lattice homomorphism between the lattices of subdiagrams of $\mathcal{M}$ and submodules of M that satisfies $\delta(\mathrm{Rad}\,\mathcal{U}) = \mathrm{Rad}\,\delta(\mathcal{U})$ ($\mathcal{U} \leq \mathcal{M}$), and $\delta(\mathcal{U})/\delta(\mathcal{V}) \cong S_{\lambda(x)}$ whenever $x \notin \mathcal{V}$ and $\mathcal{V} \cup \{x\} = \mathcal{U}$ are subdiagrams of $\mathcal{M}$.

A *potential algebra diagram* is an n-diagram $\mathcal{R}$ such that : (i) $\mathcal{R} = \mathcal{P}_1 \dot{\cup} \cdots \dot{\cup} \mathcal{P}_n$ where $\mathcal{P}_i \cap \mathcal{P}_j = \{0\}$ if $i \neq j$ and $\mathcal{P}_i = \mathcal{U}(e_i)$ with $\lambda(e_i) = i$, $i = 1,\dots, n$; and (ii) if $e_i \to a \neq 0$ in $\mathcal{R}$ then there is no other arrow $x \to a$ in $\mathcal{R}$ and there is an epimorphism (i.e., a surjective homomorphism of n-diagrams) $\phi: \mathcal{P}_{\lambda(a)} \twoheadrightarrow \mathcal{U}(a)$. In such a diagram we denote $\mathcal{J} = \mathrm{Rad}(\mathcal{R})$, $\mathcal{J}^2 = \mathrm{Rad}(\mathcal{J})$, etc.. If $\{\phi_a | a \in \mathcal{J}\backslash\mathcal{J}^2\}$ is a set of epimorphisms $\phi_a : \mathcal{P}_{\lambda(a)} \twoheadrightarrow \mathcal{U}(a)$, we let $\mathcal{H} = \mathcal{H}(\{\phi_a | a \in \mathcal{J}\backslash\mathcal{J}^2\})$ be the subsemigroup of $\mathrm{End}(\mathcal{R})$ generated by $\{\phi_a | a \in \mathcal{J}\backslash\mathcal{J}^2\}$ and the projections $\varepsilon_i : \mathcal{R} \to \mathcal{P}_i$ and define a function

$$\Theta : \mathcal{H} \to \mathcal{R}$$

via

$$\Theta : \gamma \to \gamma(e_i), \ \text{if } \gamma = \gamma \circ \varepsilon_i.$$

Fuller [7] proved that Θ is always surjective and called $(\mathcal{R}, \{\phi_a | a \in \mathcal{J}\backslash \mathcal{J}^2\})$ an *algebra diagram* if Θ is injective.

A finite semigroup $\mathcal{R}$ containing 0, is called an *algebra semigroup* in case $\mathcal{R} = \{e_1,\dots, e_n\} \cup \mathcal{J}$ with $e_1,\dots, e_n$ orthogonal idempotents and $\mathcal{J}$ a nilpotent ideal such that

$$\mathscr{R} = \bigcup_{i=1}^{n} \mathscr{R} e_i = \bigcup_{i=1}^{n} e_i \mathscr{R}.$$

In [7], with such a semigroup are associated two algebra diagrams $\mathscr{R}_\ell$ and $\mathscr{R}_r$, where $x \to y \neq 0$ in $\mathscr{R}_\ell$ in case there is an $a \in \mathscr{J} \setminus \mathscr{J}^2$ such that $ax = y$ and $x \to 0$ if $\mathscr{J}x = 0$; $\phi_a(x) = xa$; and $\lambda_\ell(x) = i$ if $e_i x \neq 0$. And $\mathscr{R}_r$ is defined similarly.

Theorem 1. *Let R be a basic left artinian ring, then R is left hereditary if and only if ${}_RR$ has a module diagram $(\mathscr{R}, \delta)$ such that $(\mathscr{R} = \mathscr{P}_1 \dot{\cup} \cdots \dot{\cup} \mathscr{P}_n, \{\phi_a | a \in \mathscr{J} \setminus \mathscr{J}^2\})$ is an algebra diagram with each $\mathscr{P}_i$ a tree and each ϕ_a an isomorphism.*

Proof. $(\Rightarrow)$ Suppose that R is a basic hereditary ring. Let

$$Je_i = \bigoplus_{a \in A_i} Re_{j_a} a e_i = \bigoplus_{a \in A_i} Ra$$

where $a = e_{j_a} a e_i \neq 0$ is the image of e_{j_a} under an isomorphism

$$Re_{j_a} \cong Re_{j_a} a e_i$$

and let

$$A = A_1 \cup \cdots \cup A_n$$

(Note : some A_i's will be empty when $Je_i = 0$). Let

$$\mathscr{J} = A \cup A^2 \cup \cdots \cup A^m \cup \{0\}$$

be the subsemigroup of $(R, \cdot)$ generated by A, and if $x \in \mathscr{J} \setminus \{0\}$ let j_x and i_x be the unique elements of $\{1, \dots, n\}$ with

$$e_{j_x} x e_{i_x} \neq 0,$$

Also $A^k = \{a_k \cdots a_1 | a_i \in A\} \setminus \{0\}$ and $m+1$ = Loewy length of R. The following observations (1)-(11) entail these hypothesis :

(1) $J^k = \bigoplus_{x \in A^k} Rx.$

Proof of (1) : If $k = 1$, we are done by hypothesis.
Suppose that $k > 1$ and

$$J^{k-1} = \bigoplus_{x \in A^{k-1}} Rx.$$

Since R is hereditary,

$$Re_{j_x} \to Rx$$

via

$$re_{j_x} \mapsto re_{j_x}x$$

is an isomorphism, so recalling $Je_{j_x} = \bigoplus_{a \in A_{j_x}} Ra$ we have $Je_{j_x}x = \bigoplus_{a \in A_{j_x}} Rax$, thus

$$J^k = J\left(\bigoplus_{x \in A^{k-1}} Rx\right) = \bigoplus_{x \in A^{k-1}} Je_{j_x}x = \bigoplus_{x \in A^{k-1}}\left(\bigoplus_{a \in A_{j_x}} Rax\right) = \bigoplus_{y \in A^k} Ry$$

since $a \in A$ and $ax \neq 0$ implies $a \in A_{j_x}$.

Let

$$\mathscr{R} = \{e_1, \dots, e_n\} \cup \mathscr{J} \text{ and } A^0 = \{e_1, \dots, e_n\}.$$

(2) If $x_1, \dots, x_\ell \in \mathscr{R}\backslash\{0\}$ and $x_i \notin \mathscr{R}x_j$ for all $i \neq j$, then

$$Rx_1 + \cdots + Rx_\ell = Rx_1 \oplus \cdots \oplus Rx_\ell.$$

Proof of (2): Renumber $x_1, \dots, x_\ell$ so that $x_1, \dots, x_t \in A^k$ and $x_{t+1}, \dots, x_\ell \in \mathscr{J}^{k+1}$. By (1), we have

$$Rx_1 \oplus \cdots \oplus Rx_t \oplus Rz_1 \oplus \cdots \oplus Rz_s = J^k,$$

where $A^k = \{x_1, \dots, x_t, z_1, \dots, z_s\}$. If $i > t$, then $x_i \in \mathscr{J}^{k+1}$, so

$$x_i = a_\ell \cdots a_{k+1} a_k \cdots a_1,$$

where, since $x_i \notin \mathscr{R}x_j$ with $j \leq t$, we must have $a_k \cdots a_1 = z_j$ for some j. Thus

$$\textstyle\sum_{i=t+1}^{\ell} Rx_i \subseteq \sum_{j=1}^{s} Rz_j.$$

Now we have

$$Rx_1 + \cdots + Rx_\ell = (Rx_1 \oplus \cdots \oplus Rx_t) \oplus (Rx_{t+1} + \cdots + Rx_\ell),$$

where the latter sum is direct by induction on ℓ.

(3) If $x, y \in \mathscr{J}$ and $Rx = Ry$ then $x = y$.

Proof of (3): If $y \in A^k$ and $x \in A^\ell$ then $k = \ell$, since by (1) $Ry \not\subseteq J^{k+1}$. But then also by (1) $x = y$.

(4) If $a_k \cdots a_1 = b_\ell \cdots b_1 \neq 0$ with $a_i, b_j \in A$ then $k = \ell$ and $b_i = a_i$, $i = 1, \dots, k$.

Proof of (4): Suppose $k \geq \ell$ and induct on k. If $k = 1$, it is clear. Suppose $k > 1$, then

$$0 \neq a_k \cdots a_1 = b_\ell \cdots b_1 \in Ra_1 \cap Rb_1$$

so by definition of A, $a_1 = b_1$, so letting $e = e_{j_{a_1}}$ we have an isomorphism

$$Re \to Rea_1$$

via

$$re \mapsto rea_1.$$

But then $a_k \cdots a_2 = b_\ell \cdots b_2$ since they have the same image, so we are done by induction. (Note : if $\ell < k$ we would eventually get $a_k \cdots a_{\ell+1} = e_i$ which is impossible.)

Now let $\mathscr{R} = \mathscr{R}_\ell$, so $\mathscr{R}$ is an algebra diagram with $\mathscr{P}_i = \mathscr{R}e_i$ and $x \to y$ if and only if $ax = y$ for some $a \in A$, and $\phi_a(x) = xa$, $\lambda(x) = e_{j_x}$ (see [7]). Also we let

$$\delta(\mathscr{U}) = R\mathscr{U},\ \mathscr{U} \leq \mathscr{R},$$

so δ is a function $\delta : \mathscr{L}(\mathscr{R}) \to \mathscr{L}({}_RR)$.

(5) $\delta(\mathscr{U}) = \oplus \{Rx \mid x \in \mathscr{U}\backslash\mathrm{Rad}(\mathscr{U})\}$.

Proof of (5) : The sum is direct by (2), and clearly the sum is contained in $\delta(\mathscr{U})$. If $y \in \mathscr{U}$ then $y \in \mathscr{U}\backslash\mathrm{Rad}(\mathscr{U})$ or there is a path from x to y for some $x \in \mathscr{U}\backslash\mathrm{Rad}(\mathscr{U})$. But then $y = a_k \cdots a_1 x \in Rx$.

(6) $\delta : \mathscr{L}(\mathscr{R}) \to \mathscr{L}({}_RR)$ is an injective lattice homomorphism.

Proof of (6) : If $\mathscr{U}\backslash\mathrm{Rad}(\mathscr{U}) = \{x_1, \ldots, x_k\}$, $\mathscr{V}\backslash\mathrm{Rad}(\mathscr{V}) = \{y_1, \ldots, y_\ell\}$ and $\delta(\mathscr{U}) = \delta(\mathscr{V})$, then by (5)

$$\delta(\mathscr{U}) = Rx_1 \oplus \cdots \oplus Rx_k = Ry_1 \oplus \cdots \oplus Ry_\ell = \delta(\mathscr{V}).$$

But then by (2) each x_i is contained in some Ry_j, so since the sums are direct and the terms are indecomposable, $\ell = k$ and we can renumber so that $Rx_i = Ry_i$ $(i = 1, \ldots, k)$. But then by (3) $x_i = y_i$, so

$$\mathscr{U} = \bigcup_{i=1}^{k} \mathscr{U}(x_i) = \bigcup_{i=1}^{k} \mathscr{U}(y_i) = \mathscr{V},$$

and δ is injective. Clearly we have

$$\delta(\mathscr{U} \cup \mathscr{V}) = \delta(\mathscr{U}) + \delta(\mathscr{V}) \text{ and } \delta(\mathscr{U} \cap \mathscr{V}) \subseteq \delta(\mathscr{U}) \cap \delta(\mathscr{V}).$$

To show " $\supseteq$ ", we write (using (2))

$$\mathscr{U} = \mathscr{U}(x_1) \dot{\cup} \cdots \dot{\cup} \mathscr{U}(x_k) \text{ and } \mathscr{V} = \mathscr{U}(y_1) \dot{\cup} \cdots \dot{\cup} \mathscr{U}(y_\ell).$$

We proceed by induction on $k + \ell$. If $k + \ell = 2$, then $\mathscr{U} = \mathscr{U}(x_1)$, $\mathscr{V} = \mathscr{U}(y_1)$ and $\mathscr{U} \subseteq \mathscr{V}$, $\mathscr{V} \subseteq \mathscr{U}$ or $R\mathscr{U} \cap R\mathscr{V} = \{0\}$ by (2), and in either case $R\mathscr{U} \cap R\mathscr{V} \subseteq R(\mathscr{U} \cap \mathscr{V})$. Suppose $k + \ell > 2$. If $x_i \notin \mathscr{U}(y_j)$ and

$y_i \notin \mathscr{U}(x_j)$ for all i and j, then by (2)

$$R\mathscr{U} \cap R\mathscr{V} = \{0\} \subseteq R(\mathscr{U} \cap \mathscr{V}).$$

Thus we may assume $x_1, \ldots, x_t \in \mathscr{V}$ for some $t \geq 1$. Then letting

$$\mathscr{U}' = \mathscr{U}(x_1) \dot{\cup} \cdots \dot{\cup} \mathscr{U}(x_t) \subseteq \mathscr{V} \text{ and } \mathscr{U}'' = \mathscr{U}(x_{t+1}) \dot{\cup} \cdots \dot{\cup} \mathscr{U}(x_k)$$

using modularity we have

$$\begin{aligned} R\mathscr{U} \cap R\mathscr{V} &= (R\mathscr{U}' + R\mathscr{U}'') \cap R\mathscr{V} \subseteq R\mathscr{U}' + (R\mathscr{U}'' \cap R\mathscr{V}) \\ &\subseteq R\mathscr{U}' + R(\mathscr{U}'' \cap \mathscr{V}) \subseteq R(\mathscr{U} \cap \mathscr{V}) \end{aligned}$$

by induction.

(7) $|\mathscr{R} \backslash \{0\}| = c({}_RR)$.

Proof of (7): This follows from (1).

(8) $\delta(\text{Rad}\,\mathscr{U}) = \text{Rad}(\delta(\mathscr{U}))$.

Proof of (8): By [7, Lemma 2.3], if $\mathscr{U} \leq \mathscr{R}$ then $\text{Rad}(\mathscr{U}) = \mathscr{J}\mathscr{U}$. Thus

$$\delta(\text{Rad}\,\mathscr{U}) = R\mathscr{J}\mathscr{U} = J\mathscr{U} = JR\mathscr{U} = \text{Rad}(\delta(\mathscr{U})).$$

(9) If $\mathscr{U} = \mathscr{V} \cup \{x\} \leq \mathscr{R}$, $x \notin \mathscr{V}$ then $\delta(\mathscr{U})/\delta(\mathscr{V}) \cong S_{\lambda(x)}$.

Proof of (9):
$$\begin{aligned} R(\mathscr{V} \cup \{x\})/R\mathscr{V} &= R\mathscr{V} + Rx/R\mathscr{V} \cong Rx/Rx \cap R\mathscr{V} \\ &= Rx/R(\mathscr{U}(x) \cap \mathscr{V}) \\ &= Rx/R(\text{Rad}(\mathscr{U}(x))) \\ &= Rx/\text{Rad}(Rx) \cong S_{\lambda(x)}. \end{aligned}$$

(10) $(\mathscr{R}, \delta)$ is a module diagram for ${}_RR$.

Proof of (10): By (6), (7), (8), and (9).

(11) Each $\mathscr{P}_i$ is a tree and each ϕ_a is an isomorphism.

Proof of (11): If $x \xrightarrow{a} z \xleftarrow{b} y$ in $\mathscr{R}$ then $ax = by$. But by (4) the latter implies $x = y$ and $a = b$. If $\phi_a(x) = \phi_a(y)$ then $xa = ya$. Again by (4), $x = y$.

($\Leftarrow$) Assuming the condition, we need only show that if $\delta(\mathscr{P}_i) = P_i \leq {}_RR$ then JP_i is projective. Since $\mathscr{P}_i$ is a tree

$$\text{Rad}(\mathscr{P}_i) = \mathscr{U}(a_1) \dot{\cup} \cdots \dot{\cup} \mathscr{U}(a_k),$$

so

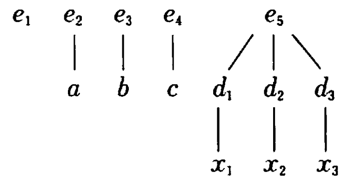
$$\begin{aligned} JP_i &= \delta(\text{Rad}\,\mathscr{P}_i) = \delta(\mathscr{U}(a_1)) \oplus \cdots \oplus \delta(\mathscr{U}(a_k)), \\ &\delta(\mathscr{U}(a_i))/\delta(\text{Rad}\,\mathscr{U}(a_i)) \cong S_{\lambda(a_i)}, \end{aligned}$$

and $P_{\lambda(a_i)}$ maps onto $\delta(\mathscr{U}(a_i))$. But $\mathscr{U}(a_i)$ is a diagram for $\delta(\mathscr{U}(a_i))$ by [7, Proposition 1.1], so

$$\begin{aligned} c(\delta(\mathscr{U}(a_i))) &= \mathrm{card}(\mathscr{U}(a_i))-1 \\ &= \mathrm{card}(\mathscr{P}_{\lambda(a_i)})-1 \quad (\text{since } \phi_a \text{ is injective}) \\ &= c(P_{\lambda(a_i)}), \end{aligned}$$

and $\delta(\mathscr{U}(a_i)) \cong P_{\lambda(a_i)}$ and JP_i is projective.

Finite global dimension is not enough to insure the existence of a diagram for an artinian ring. To see this, let $\mathscr{R}_\ell$ be the diagram

$\mathscr{R}_\ell$:


with $\lambda(a) = \lambda(b) = \lambda(c) = \lambda(x_i) = 1$ and $\lambda(d_i) = i+1$, $i = 1, 2, 3$. If K is any field, then $\mathscr{R}_\ell$ is the left diagram for the hereditary diagram algebra $K\mathscr{R}$ [7, Theorems 3.3, 4.3]. Now let S be the factor ring of R defined by

$$S = R/I \text{ with } I = K(x_1+x_2+x_3).$$

Then ${}_SS$ has a decomposition ${}_SS = Q_1 \oplus \cdots \oplus Q_5$ where $Q_i = (Re_i+I)/I$, $i = 1, \ldots, 5$; and gl.dim. $S = 2$. Let

$$L = \mathrm{Rad}(Q_5) \text{ and } N = \mathrm{Rad}(L).$$

Then L is a sum of three uniserial modules, each of length 2,

$$L = D_1+D_2+D_3$$

with socles $X_i = \mathrm{Soc}(D_i) \cong Q_1$, $i = 1, 2, 3$, such that

$$L/N = \oplus_{i=1}^{3}(D_i+N)/N \cong \oplus_{i=1}^{3} Q_{i+1}/\mathrm{Rad}(Q_{i+1})$$

is a direct sum of three non-isomorphic simples, and

$$N = X_1 \oplus X_2 = X_1 \oplus X_3 = X_2 \oplus X_3 .$$

One can show that, since S is an artinian ring, if ${}_SS$ has a diagram then so does (the radical of) every indecomposable projective left S-module (we don't know whether diagrams are inherited by submodules, or even by direct summands, in general). So suppose that $(\mathscr{L}, \delta)$ is a diagram for L. Then $N = \delta(\mathrm{Rad}\,\mathscr{L})$, and since the $(D_i+N)/N$ are the only simple submodules of L/N, there are subdiagrams $\mathscr{U}_1$, $\mathscr{U}_2$, $\mathscr{U}_3$ of $\mathscr{L}$ such that

$$\delta(\mathscr{U}_i) = D_i+N, \qquad i = 1, 2, 3.$$

But then, letting $\mathscr{X}_i = \mathrm{Rad}(\mathscr{U}_i)$, we have

$$\delta(\mathscr{X}_i) = \mathrm{Rad}(D_i+N) = X_i, \quad i = 1, 2, 3,$$

which is imposible because $\mathrm{Card}((\mathscr{X}_1 \cup \mathscr{X}_2 \cup \mathscr{X}_3)\setminus\{0\}) > 2 = c(N)$. (The module L appeared in [5], and Doug Pickering pointed out that it could not have a diagram of the type considered here.)

Camillo, Fuller and Haack [6] noted that if C and D are division rings then by [6, Lemma 2. 1] a bi-vector space ${}_CV_D$ is exact if and only if it has a (left or, equivalently, right) basis $v_1, \ldots, v_n$ (which they called an exact basis) such that

$$\textstyle\sum_{i=1}^{k} Cv_i = \sum_{i=1}^{k} v_iD, \qquad k = 1, \ldots, n.$$

They proved that an artinian ring R with $J^2 = 0$ is exact if and only if for each pair of primitive idempotents e and f in R, the bi-vector space ${}_{\overline{eRe}}eJf_{\overline{fRf}}$ is exact or zero, where $\overline{eRe} = eRe/eJe$ and $\overline{fRf} = fRf/fJf$. Using these facts, the following result is immediate.

Lemma 2. *Let R be a basic exact artinian ring with $J^2 = 0$, then there is a subset $\Lambda \subseteq J$ such that $J = \bigoplus_{a\in\Lambda} Ra = \bigoplus_{a\in\Lambda} aR$ with all Ra and aR simple modules.*

According to Theorem 1 and its proof, a hereditary artinian ring R has a left diagram $(\mathscr{R}_\ell, \delta_\ell)$ induced from an algebra semigroup $\mathscr{R}$ and a right diagram $(\mathscr{R}'_r, \delta'_r)$ induced from an algebra semigroup $\mathscr{R}'$. These semigroups are subsemigroups of $(R, \cdot)$. The next theorem shows that if R is also exact, then $(R, \cdot)$ contains an algebra semigroup which induces both a left and a right diagram for R.

Theorem 3. *Let R be a basic exact artinian ring. If R is hereditary, there is an algebra semigroup $\mathscr{R}$ such that $(\mathscr{R}_\ell, \delta_\ell)$ is a diagram for ${}_RR$, and $(\mathscr{R}_r, \delta_r)$ is a diagram for R_R, where $\delta_\ell(\mathscr{U}) = R\mathscr{U}$, $\mathscr{U} \leq \mathscr{R}_\ell$, and $\delta_r(\mathscr{U}) = \mathscr{U}R$, $\mathscr{U} \leq \mathscr{R}_r$.*

Proof. If R is hereditary, then

$$J = \bigoplus_{a \in \mathrm{A}} Ra \text{ if and only if } J/J^2 = \bigoplus_{a \in \mathrm{A}} (Ra+J^2)/J^2.$$

So by Lemma 2, there is an $\mathrm{A} \subseteq J$ such that

$$J = \bigoplus_{a \in \mathrm{A}} Ra = \bigoplus_{a \in \mathrm{A}} aR, \text{ where each } a = e_{j_a} a e_{i_a} \neq 0.$$

Let $\mathrm{m}+1$ = Loewy length of R and

$$\mathscr{J} = \mathrm{A} \cup \mathrm{A}^2 \cup \cdots \cup \mathrm{A}^{\mathrm{m}} \cup \{0\}.$$

According to the proof of Theorem 1,

$$\mathscr{R} = \{e_1, \ldots, e_n\} \cup \mathscr{J},$$

is a semigroup that does the job.

We call the algebra semigroup $\mathscr{R}$ of Theorem 3 an *algebra semigroup for R*.

Habeb [8] proved that the endomorphism of the minimal cogenerator over a basic exact ring R is an exact ring. This result answers Azumaya's conjecture partially. If R is also hereditary, we can say more about these two rings.

Theorem 4. *Let R be a basic exact artinian ring with $_RE$ the minimal cogenerator and $S = \mathrm{End}(_RE)$. If R is hereditary, then R and S have the same left and right algebra diagrams.*

Proof. Let $J = J(R)$, $N = J(S)$, and $f_i \in S$ be the idempotents for $E_i = E(Re_i/Je_i)$ in the decomposition $_RE = E_1 \oplus \cdots \oplus E_n$ of the minimal cogenerator. One checks that $_RE_S$ defines a duality such that E_i corresponds to f_iS and Re_i/Je_i corresponds to f_iS/f_iN. Using these correspondences and [4, Corollary 3 and Corollary 5] we have

$$\mathrm{c}((f_iSf_j)_{f_jSf_j}) = \mathrm{c}((_{e_jRe_j}(e_jE_i)) = \mathrm{c}((e_iRe_j)_{e_jRe_j})$$

for each i and j. It follows that $(e_iR/e_iJ^2)_R$ and $(f_iS/f_iN^2)_S$ have the same diagrams since $_{R/J^2}(\mathrm{r}_E(J^2))_{S/N^2}$ also defines a duality.(Using the fact $\mathrm{r}_E(J) = \ell_E(N)$, one easily shows that $\mathrm{r}_E(J^2) = \ell_E(N^2)$, so this duality follows from [10, p. 1345].) Thus since R and S are hereditary it follows that $e_iJ \cong \bigoplus_j(e_jR)^{k_{ij}}$ if and only if $f_iN \cong \bigoplus_j(f_jS)^{k_{ij}}$. Now by induction on $\mathrm{c}(e_iR)$ we see that the right (similarly, the left) diagrams of R and S are the same.

Corollary 5. *Let R be a basic exact hereditary artinian ring with ${}_RE$ the minimal cogenerator, then R and $S = \mathrm{End}({}_RE)$ have isomorphic algebra semigroups.*

Proof. Let $A_{ij} \subseteq e_iJe_j \backslash J^2$ with $\{a+J^2 | a \in A_{ij}\}$ an exact basis for $(e_iJe_j+J^2)/J^2$, and let $A = \bigcup_{i,j} A_{ij}$. If $m+1$ = the Loewy length of R, then

$$\mathscr{R} = \{e_1, \dots, e_n\} \cup A \cup A^2 \cup \cdots \cup A^m \cup \{0\}$$

is an algebra semigroup for R. Now since S is also a basic exact hereditary artinian ring which has the same diagram as R, we have the corresponding B_{ij}, $B = \bigcup_{i,j} B_{ij}$, and

$$\mathscr{S} = \{f_1, \dots, f_n\} \cup B \cup B^2 \cup \cdots \cup B^m \cup \{0\}$$

is an algebra semigroup for S, where $|A_{ij}| = |B_{ij}|$ for each i and j. Let

$$\varphi_{ij} : A_{ij} \to B_{ij}$$

be any bijection. Then the φ_{ij}'s induce a bijection φ from A to B. Define a map

$$\beta : \mathscr{R} \to \mathscr{S}$$

via $\beta(e_i) = f_i$, $\beta(0) = 0$, and $\beta(a_1 \cdots a_k) = \varphi(a_1) \cdots \varphi(a_k)$, where $a_1 \cdots a_k \in A^k$. From (4) in the proof of Theorem 1, it follows that β is well-defined and hence a semigroup homomorphism. Now β is onto and $|R| = |S|$ is finite, hence β is a semigroup isomorphism.

Our concluding proposition characterizes exactness for hereditary artinian rings via exact bases.

Proposition 6. *Let R be an indecomposable artinian ring with complete set of primitive idempotents $e_1, \dots, e_n$ such that each e_iRe_i is a division ring (e.g., a hereditary artinian ring). Then R is exact if and only if R has an exact basis over a division subring D such that $De_i = e_iRe_i = e_iD$, $i = 1, \dots, n$.*

Proof. Let $R_i = e_iRe_i$ be a division ring, $i = 1, \dots, n$. If R is exact then by [6, Proposition 2.2] there are isomorphisms $\phi_i : R_1 \to R_i$ $i = 2, \dots, n$; and by [6, Lemma 1.4] each bi-vector space ${}_{R_i}(e_iRe_j)_{R_j}$ is exact and so has an exact basis. Now letting

$$D = \{r_1 + \phi_2(r_1) + \cdots + \phi_n(r_1) \mid r_1 \in R_1\},$$

the union of these exact bases becomes one for ${}_DR_D$.

Conversely, if D satisfies the condition and $v_1, \ldots, v_m$ is an exact basis for ${}_DR_D$ then $\{e_i v_1 e_j, \ldots, e_i v_m e_j\}$ contains an exact basis for ${}_{R_i}(e_i R e_j)_{R_j}$, and R is exact by [6, Proposition 2.7 and Theorem 2.5].

Using the inexact real bi-vector space ${}_{\mathbb{R}}V_{\mathbb{R}}$ of [6, Example 2.9] we note that the hereditary ring

$$\begin{bmatrix} \mathbb{R} & V \\ 0 & \mathbb{R} \end{bmatrix}$$

satisfies $De_i = e_i R e_i = e_i D$, but is not exact.

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(*Received January 25, 1988*)