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SOME DECOMPOSITION THEOREMS FOR RINGS

Dedicated to Professor Miyuki Yamada on his 60th birthday

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Throughout, R will represent a ring with center C. An element x of R is called a *right p.p. element* if there exists an idempotent e in R such that $xe = x$ and $\{r \in R \mid xr = 0\} = \{r \in R \mid er = 0\}$ (see [5]). Let P be the set of right p.p. elements in R, and N the set of nilpotent elements in R. Following [3], R is called a P_k'*-ring* if $xR^k = xR^kx$ for all $x \in R$; following [7], R is called a J_{k+1}*-ring* if for each $x_1, \cdots, x_{k+1} \in R$ there exists an integer $n > 1$ such that $x_1\cdots x_{k+1} = (x_1\cdots x_{k+1})^n$, where k is a fixed positive integer.

The main theorems of the present paper are stated as follows :

Theorem 1. (1) *The following are equivalent :*

1) *For each sequence* $\{x_i\}_{i\in N}$ *of elements in* R *there exist positive integers* k, k' *such that* $x_1\cdots x_{k+1} = x_1^{k+2}z$ *and* $x_{k'+1}\cdots x_1 = x_1^{k'+2}z'$ *with some* $z, z' \in R$.

2) *For each sequence* $\{x_i\}_{i\in N}$ *of elements in* R *there exist positive integers* k, k' *such that* $x_1\cdots x_{k+1} = (x_1\cdots x_{k+1})^2z$ *and* $x_{k'+1}\cdots x_1 = (x_{k'+1}\cdots x_1)^2z'$ *with some* $z, z' \in R$.

3) $R = N \oplus P$ *(strictly speaking, both* N *and* P *are ideals of* R *and* R *is the direct sum of* N *and* P*),* P *is strongly regular, and* N *is (right and left) T-nilpotent.*

(2) *Let* k *be a positive integer. Then the following are equivalent :*

1) R *is a* P_k'*-ring.*

2) *For each* $x_1, \cdots, x_{k+1} \in R$ *there exists* $z \in R$ *such that* $x_1\cdots x_{k+1} = (x_1\cdots x_{k+1})^2z$.

3) $R = N \oplus P$, P *is strongly regular, and* $N^{k+1} = 0$.

(3) *Let* k *be a positive integer. Then the following are equivalent:*

1) R *is a* J_{k+1}*-ring.*

2) *For each* $x_1, \cdots, x_{k-1} \in R$ *there exists* $z \in \langle x_1, \cdots, x_{k+1}\rangle$ *such that* $x_1\cdots x_{k+1} = (x_1\cdots x_{k+1})^2z$.

3) $R = N \oplus P$, P *is a J-ring, and* $N^{k-1} = 0$.

Theorem 2. *Let k be a positive integer, and A an additive subsemigroup of R with $A \subseteq N$. Suppose that for each $x_1, \cdots, x_{k+1} \in R \setminus A$ there exists $z \in R$ such that $x_1 \cdots x_{k+1} = (x_1 \cdots x_{k+1})^2 z$. Then $R = N \oplus P$, P is strongly regular, and $N^{k+1} = 0$.*

Proof of Theorem 1. (1) We can easily see that 3) implies 1) and 2).

1) $\Rightarrow$ 3). Let $\{x_i\}_{i \in N}$ be an arbitrary sequence of elements in R with $x_1 \in N$. Then there exists $k_1 > 0$ such that $x_1 \cdots x_{k_1+1} = x_1^{k_1+2} z_1$ with some $z_1 \in R$. Next, there exists $k_2 > 0$ such that $x_1^{k_1+2} z_1 x_{k_1+2} \cdots x_{k_1+k_2} = x_1^{(k_1+2)(k_2+2)} z_2$ with some $z_2 \in R$, and so $x_1 \cdots x_{k_1+k_2} = x_1^{(k_1+2)(k_2+2)} z_2$. Continuing the same procedure, we obtain eventually a positive integer k such that $x_1 \cdots x_{k+1} = 0$; similarly, we can find a positive integer k' such that $x_{k'+1} \cdots x_1 = 0$. In particular, for any idempotent e of R we have $Ne = eN = 0$. Hence e is central, and so all the idempotents of R generate a reduced ideal. Furthermore, for each $x \in R$ there exists $h > 0$ such that $x^{h+1} \in x^{h+2} R$, and so R is right π-regular. Hence, by a result of Zöschinger-Dischinger(see, e.g., [4, Proposition 2]), R is strongly π-regular. Now, by [5, Corollary 1], $R = N \oplus P$, P is strongly regular, and N is T-nilpotent.

2) $\Rightarrow$ 3). We claim first that every idempotent e of R belongs to C. Actually, for any $x \in R$ there exist $z, z' \in R$ such that $(x - ex)e = ((x - ex)e)^2 z = 0$ and $e(x - xe) = (e(x - xe))^2 z' = 0$, and therefore $xe = exe = ex$.

For each $x \in R$ there exists a positive integer k such that $x^{k+1} \in x^{2(k+1)} R$. Hence, again by a result of Zöschinger-Dischinger, R is strongly π-regular. Now, let e be an arbitrary (central) idempotent of R, and $a \in N$: $a^n = 0$. Then $ea \in (ea)^2 R = (ea)^n R = a^n e R = 0$, and therefore $eR \cap N = 0$. Hence, by [5, Corollary 1], $R = N \oplus P$ and P is strongly regular. If $\{x_i\}_{i \in N}$ is an arbitrary sequence of elements in R such that x_1 is in the nil ideal N, then we can easily see that there exist positive integers k, k' such that $x_1 \cdots x_{k+1} = x_{k'+1} \cdots x_1 = 0$.

(2) The equivalence of 1) and 3) has been proved in [5, Corollary 2]. Careful scrutiny of the proof of (1) shows that 2) and 3) are equivalent.

(3) Obviously, 3) $\Rightarrow$ 1) $\Rightarrow$ 2).

2) $\Rightarrow$ 3). By (2), $R = N \oplus P$, P is strongly regular, and $N^{k+1} = 0$. For any $x \in R$ there exists $z \in \langle x \rangle$ such that $x^{k+1} = x^{2(k+1)} z$. Hence R is

periodic by Chacron's criterion [2], and so P is a J-ring. (This implication is also an easy consequence of [6, Theorem 1].)

Theorem 1 (3) improves [1, Theorem 1] as well as [7, Theorem 2.4], and enables us to generalize [1, Theorem 2] as follows :

Corollary 1. *Let k be a positive integer. If for each $x_1,\cdots,x_{k+1}\in R$ there exists $z\in\langle x_1,\cdots,x_{k+1}\rangle$ such that $x_1\cdots x_{k+1}=(x_{k+1}\cdots x_1)^2z$, then $R=N\oplus P$, P is a J-ring, and $N^{k+1}=0$.*

Proof. There exists $z'\in\langle x_1,\cdots,x_{k+1}\rangle$ such that $x_{k+1}\cdots x_1=(x_1\cdots x_{k+1})^2z'$. Hence $x_1\cdots x_{k+1}=(x_1\cdots x_{k+1})^2z'(x_{k+1}\cdots x_1)z$, and the assertion is clear by Theorem 1 (3)

Proof of Theorem 2. As is easily seen,

($*$) if $x_1,\cdots,x_{k+1}\in R\backslash A$ and $x_1\cdots x_{k+1}\in N$ then $x_1\cdots x_{k+1}=0$; in particular, if $b\in N\backslash A$ then $b^{k+1}=0$.

Next, we claim that

($**$) if $b\in N\backslash A$ and $x\in R$ then $bx\in N$ and $xb\in N$.

In order to see this, we may assume $bx\notin A$. Noting that $b^{k+1}=0$, we see that $b^{k-1}bxb\in N$, and so $b^{k-1}bxb=0$ by ($*$). Hence $b^{k-1}(bx)^2=0$. Next, by making use of the fact that $b^{k-2}(bx)^2b\in N$ we can see that $b^{k-2}(bx)^3=0$. Continuing the same procedure, we obtain eventually $(bx)^{k+1}=0$. Hence $bx\in N$; similarly, $xb\in N$.

Now, let $x_1,\cdots,x_{k+1}\in R\backslash A$ and $a_1,\cdots,a_{k+1}\in A$. If x_i is in N then $x_i\cdots x_{k+1}x_1\cdots x_{i-1}=0$ by ($**$) and ($*$), and therefore $x_1\cdots x_{k+1}=0$, again by ($**$) and ($*$) ; namely $(R\backslash A)^{i-1}(N\backslash A)(R\backslash A)^{k+1-i}=0$. Noting that $x_j-a_j\in R\backslash A$ $(j\neq i)$, we get $x_1\cdots a_j\cdots x_{k+1}=x_1\cdots x_{k+1}-x_1\cdots(x_j-a_j)\cdots x_{k+1}=0$, and repeating this procedure, we can easily see that $R^{i-1}(N\backslash A)R^{k+1-i}=0$. Furthermore, if $a_i^s=0$ then $(x_i-a_i)^{(k+1)s}=(-a_i)^{(k+1)s}=0$, and so $x_i-a_i\in N\backslash A$. Hence we can see that $R^{i-1}AR^{k+1-i}=0$ $(i=1,\cdots,k+1)$. Now the assertion is clear by Theorem 1 (2).

Corollary 2. *Let k be a positive integer, and A an additive subsemigroup of R properly contained in the Jacobson radical J of R. Suppose that for each $x_1,\cdots,x_{k+1}\in R\backslash A$ there exists $z\in R$ such that $x_1\cdots x_{k+1}=(x_1\cdots x_{k+1})^2z$. Then $R=N\oplus P$, P is strongly regular, and $N^{k+1}=0$.*

Proof. In view of Theorem 2, it suffices to show that $J^{k+1}=0$. Let $x_1,\cdots,x_{k+1}\in J\backslash A$ and $a_1,\cdots,a_{k+1}\in A$. Then there exists $z\in R$ such that $x_1 \cdots x_{k+1} = (x_1 \cdots x_{k+1})^2 z \in x_1 \cdots x_{k+1} J$. Hence $x_1 \cdots x_{k+1} = 0$; namely $(J \backslash A)^{k+1} = 0$. Since $x_i - a_i \in J \backslash A$ for all i, we can easily see that $J^{k+1} = 0$ (see the proof of Theorem 2).

Finally, the following is immediate by Theorem 2, Corollary 2 and the proof of Theorem 1 (3), and generalizes [1, Theorems 3 and 4].

Corollary 3. *Let k be a positive integer, and A an additive subsemigroup of R with $A \subsetneq N$ (or $A \subsetneq J$). Suppose that for each $x_1, \cdots, x_{k+1} \in R \backslash A$ there exists $z \in \langle x_1, \cdots, x_{k+1} \rangle$ such that $x_1 \cdots x_{k+1} = (x_1 \cdots x_{k+1})^2 z$.* Then $R = N \oplus P$, *P is a J-ring, and* $N^{k+1} = 0$.

References

[1] H. E. Bell and S. Ligh : Some decomposition theorems for periodic rings and near rings, Math. J. Okayama Univ. **31** (1989), 93–99.

[2] M. Chacron : On a theorem of Herstein, Canad. J. Math. **21** (1969), 1348–1353.

[3] K. Chiba and H. Tominaga : On generalizations of P_1-rings and gsr-rings, Math. J. Okayama Univ. **18** (1976), 149–152.

[4] Y. Hirano : Some studies on strongly π-regular rings, Math. J. Okayama Univ. **20** (1978), 141–149.

[5] Y. Hirano and H. Tominaga : Rings decomposed into direct sums of nil rings and certain reduced rings, Math. J. Okayama Univ. **27** (1985), 35–38.

[6] Y. Hirano, H. Tominaga and A. Yaqub : On rings in which every element is uniquely expressible as a sum of a nilpotent element and a certain potent element, Math. J. Okayama Univ. **30** (1988), 33–40.

[7] M. Ó Searcóid : A structure theorem for generalized J-rings, Proc. Roy. Irish Acad. **87A** (1987), 117–120.

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