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SOME DECOMPOSITION THEOREMS FOR PERIODIC RINGS AND NEAR-RINGS

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A recent note in the American Mathematical Monthly [10] establishes commutativity of a ring R satisfying the condition $(xy)^{n(x,y)} = xy$ with $n(x, y) > 1$. More recently, S. Ligh and J. Luh [7] have given a proof that such rings are direct sums of J-rings (i.e. rings satisfying Jacobson's $x^n = x$ property) and zero rings. It is natural to consider the related properties $xy = (xy)^2p(xy)$ or $xy = (yx)^2p(yx)$, where $p(X) \in Z[X]$; and H. Tominaga and A. Yaqub [11, Theorem 2] have obtained a commutativity result under such hypotheses. These theorems are the jumping-off point for the work in Section 1, which considers slightly more general polynomial-constraint conditions and establishes direct-sum decomposition theorems. Section 2 deals with some related problems for near-rings.

1. Some decomposition theorems for rings. For the purposes of this section, R will denote a ring and N its set of nilpotent elements. The set of potent elements — that is, $\{x \in R \mid x^n = x$ for some $n > 1\}$ — will be denoted by P. The symbol Z will denote the ring of integers, $Z[X]$ the ring of polynomials in one indeterminate and $Z\langle X, Y\rangle$ the ring of polynomials in two noncommuting indeterminates.

The ring R is called periodic if for each $x \in R$ there exist distinct positive integers $m = m(x)$ and $n = n(x)$ for which $x^m = x^n$. A sufficient condition for R to be periodic is Chacron's criterion: for each $x \in R$, there exists a positive integer m and a polynomial $p(X) \in Z[X]$ such that $x^m = x^{m+1}p(x)$ ([6], [3, Theorem 1]).

Theorem 1. *Suppose that for each $x, y \in R$, there exists $p(X, Y) \in Z\langle X, Y\rangle$ such that*

$$(*) \qquad xy = (xy)^2p(x, y).$$

Then R is a direct sum of a J-ring and a zero ring.

Proof. Taking $x = y$ in (*), we get $q(X) \in Z[X]$ for which $x^2 =$

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$x^4 q(x)$; hence R is periodic by Chacron's criterion, and by [1, Lemma 1 (c)], $R = P+N$. It is clear from (*) that R is 0-commutative — i.e. $xy = 0$ implies $yx = 0$; hence, as is easily verified, N is an ideal. Since (*) yields $xy = (xy)^n (p(x, y))^{n-1}$ for all $n \geq 2$, it is now immediate that

$$RN = NR = \{0\} ; \tag{1}$$

in particular, N is a zero ring.

To complete the proof, it is sufficient to show that each element of R is uniquely representable as the sum of a potent element and a nilpotent element [5, Theorem 3]. Accordingly, suppose that $a+u = b+v$, where $a, b \in P$ and $u, v \in N$; rewrite as

$$a-b = v-u. \tag{2}$$

Choose a single odd integer $k > 1$ for which $a^k = a$ and $b^k = b$, and note that $e_1 = a^{k-1}$ and $e_2 = b^{k-1}$ are idempotents with $e_1 a = a$ and $e_2 b = b$. Left-and-right-multiplying (2) by a and b, and recalling (1), we get $a^2 = ab = ba$ and $ab = ba = b^2$; hence $a^2 = b^2$ and $e_1 = e_2$. Left-multiplying (2) by e_1 now yields $a = b$, and we are finished.

A slight modification of the proof yields

Theorem 2. *Suppose that for each $x, y \in R$, there exists $p(X, Y) \in Z\langle X, Y\rangle$ such that $xy = (yx)^2 p(x, y)$. Then R is a direct sum of a J-ring and a zero ring.*

Our next theorem, which extends Theorem 2 of [11], shows that under appropriate restrictions, it is enough to require (*) for a proper subset of R.

Theorem 3. *Let R be a ring with $N \neq \{0\}$, and let A be an additive subgroup of R with $A \subsetneqq N$. Suppose that for each $x, y \in R \backslash A$, there exists $p(X, Y) \in Z\langle X, Y\rangle$ such that (*) is satisfied. Then R is a direct sum of a J-ring and a zero ring.*

Proof. Taking $y = x \in R\backslash A$ shows that
(3) for each $x \in R\backslash A$, there exists $p(X) \in Z[X]$ such that $x^2 = x^4 p(x)$. Furthermore, since elements of A are nilpotent, for each $x \in A$ there exist m and n such that $x^m = x^{m+n}$. Once again, R is periodic by Chacron's criterion. If we can show that N is an ideal which annihilates R on both sides, we are finished ; the rest of the argument is the same as in Theorem 1. Observe that

(4) $$\text{if } x \in N\backslash A, \text{ then } x^2 = 0\,;$$

this follows directly from (3). Next we show that

(5) $$\text{if } x \in N\backslash A,\ y \in R, \text{ and } xy = 0, \text{ then } yx = 0.$$

Clearly this is the case if $y \in R\backslash A$, so we suppose $xw = 0$, where $x \in N\backslash A$ and $w \in$ A. Now $x+w \notin A$, and $x(x+w) = x^2+xw = 0$; hence $(x+w)x = 0 = x^2+wx = wx$.

Next we show that N is an ideal. If $x, y \in A$, then $x-y \in A \subseteq N$. On the other hand, if $x \in N\backslash A$, we see from (4) and (5) that

(6) $$xRx = \{0\}\,;$$

hence if $y \in N$ and $y^r = 0$, every product of $2r$ factors, each an x or y, is trivial, so that $(x-y)^{2r} = 0$. Now if $x \in N\backslash A$, (6) shows that $(xr)^2 = (rx)^2 = 0$ for all $r \in R$; and if $w \in A$, choosing $x \in N\backslash A$ and writing $wr = (x+w)r-xr$ shows that $wr \in N$. Thus N is an ideal as claimed.

If $x \in N\backslash A$ and $y \in R\backslash A$, (*) and the fact that N is an ideal imply that $xy = 0$. Thus, capitalizing again on the fact that every element of A is a difference of two elements of $R\backslash A$, we show easily that $RN = NR = \{0\}$, thereby completing the proof of Theorem 3.

Our last theorem of this section is, in the end, a corollary of Theorem 3.

Theorem 4. *Let R be a 2-torsion-free ring, J its Jacobson radical, and A an additive subgroup with $A \subsetneq J$. Suppose that (*) holds for each $x, y \in R\backslash A$. Then R is a direct sum of a J-ring and a zero ring.*

Proof. If $x \in J\backslash A$, (*) yields $q(X) \in Z[X]$ such that $x^2 = x^4q(x)$ — a property which for elements of J implies $x^2 = 0$. Thus, if $y \in A$ and $x \in J\backslash A$, we have $x^2 = 0 = (x+y)^2$, so that

(7) $$xy+yx+y^2 = 0 \text{ for all } x \in J\backslash A,\ y \in A.$$

Replacing x by $x+y$ in (7) and then subtracting (7) from the result gives $2y^2 = 0$, hence $y^2 = 0$; thus J is a nil ideal. Now $\bar{R} = R/J$ satisfies the hypotheses of Theorem 1, hence is certainly commutative. Consequently $[x, y] \in J$ for all $x, y \in R$ and $[x, y]^2 = 0$ for all $x, y \in R$. But this condition is known to imply that N is an ideal; hence $N = J$, and our conclusion follows from Theorem 3.

2. Some near-ring results. For near-rings, the analogous hypotheses do not quite yield direct-sum decompositions, so we define a weaker notion of *orthogonal sum*. Specifically, a near-ring R is an orthogonal sum of sub-near-rings A and B — denoted $R = A \dotplus B$ — if $AB = BA = \{0\}$ and each element is uniquely representable in the form $a+b$, with $a \in A$ and $b \in B$. We also define R to be 0-commutative if $xy = 0$ implies $yx = 0$. For this section, N and P are as before, and C denotes the multiplicative center of R. The symbol R' denotes the commutator subgroup of the additive group $(R, +)$.

Before stating our theorems, we present some lemmas to be used in the proofs.

Lemma 1. *Let R be a near-ring in which idempotents are multiplicatively central. If e and f are any idempotents, there exists an idempotent g such that $ge = e$ and $gf = f$.*

Proof. Note that $R = eR \dotplus A(e)$, where $A(e)$ denotes the annihilator of e; and write $f = f_1+f_2$, where $f_1 \in eR$ and $f_2 \in A(e)$. In view of the uniqueness-of-representation built into the definition of orthogonal sum, it is easy to show that f_1 and f_2 are idempotents. Take g to be $e+f_2$. Then

$$g^2 = (e+f_2)e+(e+f_2)f_2 = e(e+f_2)+f_2(e+f_2) = e+f_2 = g.$$

Moreover, $ge = (e+f_2)e = e(e+f_2) = e$ and

$$gf = (e+f_2)f_1+(e+f_2)f_2 = f_1(e+f_2)+f_2(e+f_2) = f_1+f_2 = f.$$

Lemma 2. *If R is a 0-commutative periodic near-ring, then $R = N+P$.*

Proof. The 0-commutativity allows us to adapt the proof in [1, Lemma 1].

The remaining lemma, to be given without proof, simply collects results which are readily accessible in the literature.

Lemma 3. (a) [9, Corollary 1.8] *If R is a 0-commutative near-ring, then N is an ideal.*

(b) [3, Theorem 12] *Let R be a distributively-generated (d-g) near-ring such that for each $x \in R$, there exist a positive integer $n = n(x)$ and an element u in the sub-near-ring generated by x, for which $x^n = x^n u$. If $N \subseteq$*

C, then R is periodic and commutative.

(c) [2, Lemma 2(2)]. *If R is a periodic d-g near-ring and $N \subseteq C$, then $R' \subseteq N$.*

(d) [4, Theorem 1] *If R is a periodic near-ring with 1 and $N \subseteq C$, then $(R, +)$ is abelian.*

Theorem 5. *Let R be a d-g near-ring with the property that for each $x, y \in R$,*

$$(\dagger) \qquad xy = p(xy),$$

where $p(xy)$ denotes an element of R which is a finite sum of powers $(xy)^k$, $k \geq 2$, and additive inverses of such powers. Then R is periodic and commutative. Moreover, $R = N \dot{+} P$, where N is a near-ring with trivial multiplication and P is a J-ring.

Proof. It is clear that R is 0-commutative, hence N is an ideal by Lemma 3(a). It follows from $(\dagger)$ that $NR = RN = \{0\}$, so that $N \subseteq C$ and $N^2 = \{0\}$. Taking $y = x$ in $(\dagger)$ gives an element r in the sub-near-ring generated by x such that $x^2 = x^2 r$, hence R is periodic and commutative by Lemma 3(b). From Lemma 2, we now know that $R = N + P$.

It remains to show that (i) P is a J-ring and (ii) each element of R has at most one representation in the form $u+a$ with $u \in N$ and $a \in P$. Proceeding to (i), let $a, b \in P$ and choose $k > 1$ such that $a^k = a$ and $b^k = b$. Then $e = a^{k-1}$ and $f = b^{k-1}$ are idempotents such that $ea = a$ and $fb = b$. Obviously $(ab)^k = a^k b^k = ab$, hence $ab \in P$. Moreover, since R/N has the $x^n = x$ property, we have $j > 1$ such that

$$(**) \qquad (a-b)^j = a-b+u, \; u \in N.$$

Using Lemma 1, choose an idempotent g for which $ge = e$ and $gf = f$; and noting that $ga = a$ and $gb = b$, multiply (**) by g, obtaining $(a-b)^j = a-b$ — that is, $a-b \in P$. Since $R' \subseteq N$ by Lemma 3(c), we now have $a+b-a-b \in P \cap N = \{0\}$; hence $(P, +)$ is abelian and P is a J-ring.

To establish (ii), suppose that $u+a = v+b$, where $u, v \in N$ and $a, b \in P$. Then $-v+u = b-a \in P \cap N = \{0\}$, hence $a = b$ and $u = v$.

Theorem 6. *Let R be a near-ring such that for each $x, y \in R$, there exists $n = n(x, y) > 1$ such that*

$$(\dagger\dagger) \qquad xy = (yx)^n.$$

Then N is a near-ring with trivial multiplication, P is a sub-near-ring with $(P, +)$ abelian, and $R = N \dotplus P$.

Proof. Since R is obviously 0-commutative, N is an ideal by Lemma 3(a); and it follows that $RN = NR = \{0\}$, and hence that $N \subseteq C$. Taking $y = x$ in (††) shows that R is periodic, and we conclude from Lemma 2 that $R = N + P$.

Now look at a typical idempotent e. For $x \in R$, we have $n, m > 1$ such that $ex = (xe)^m$ and $xe = (ex)^n$. Right-multiplying the first of these by e and left-multiplying the second by e, we get $ex = exe = xe$; thus, idempotents are central.

Next we show that P is a sub-near-ring and $(P, +)$ is abelian. That P is closed under addition is shown as in the proof of Theorem 5, and a similar argument yields multiplicative closure. To show that $(P, +)$ is abelian, let $a^k = a$ and $b^k = b$, $k > 1$; and for the idempotents $e = a^{k-1}$ and $f = b^{k-1}$, let g be an idempotent such that $ge = e$ and $gf = f$. Now gR is a periodic near-ring with multiplicative identity element whose nilpotent elements are central, hence $(gR, +)$ is abelian by Lemma 3(d). Therefore $ga + gb - ga - gb = 0$; and since $ga = a$ and $gb = b$, we have $a + b - a - b = 0$.

Finally, we note that the uniqueness-of-representation argument is as for Theorem 5.

Example. From the list of near-rings R with additive group S_3, consider Example 29 on p. 342 of [8]. This near-ring is commutative and satisfies the identity $xy = (xy)^2$, but P is not an ideal. Thus, we cannot get a direct-sum decomposition under the hypotheses of Theorem 3 or Theorem 6.

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