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A NOTE ON FIXED RINGS

Dedicated to Professor Hiroyuki Tachikawa on his 60th birthday

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Let A be an associative ring with identity and G a finite group acting as automorphisms on A. Let A^G denote the fixed subring of A consisting of the elements left fixed by every element of G.

In [3], Handelman and Renault showed that for a certain class of rings, A is finitely generated projective over A^G. In this note we consider the question of when the ring extension A of A^G is a Frobenius one in the sense of Kasch [4], and a characterization of A to be a Frobenius extension of A^G having the trace $\mathrm{tr} : A \to A^G$ defined by $\mathrm{tr}(a) = \sum_{\sigma \in G} \sigma(a)$ as a Frobenius map is obtained by using the trivial crossed product of A relative to G. As a consequence, we can give another proof of our previous result of [5] and sharpen Theorem 4 of [3] slightly.

Throughout this paper A denotes an associative ring with identity 1. The notation M_A (resp. ${}_AM$) denotes M a unital, right (resp. left) A-module.

Let B be a subring of A containing the same identity 1. According to Kasch [4], a ring extension A/B is called a Frobenius one if A is finitely generated projective as a right B-module and $A \simeq \mathrm{Hom}(A_B, B_B)$ as B-A-bimodules. The conditions are equivalent to the existence of a B-B-homomorphism $h : A \to B$ and a finite number of elements r_i's, l_i's in A such that $x = \sum_i r_i h(l_i x) = \sum_i h(x r_i)\, l_i$ for all $x \in A$ ([8]). When this is the case, we shall call a system $(h\,;\, r_i, l_i)_i$ a Frobenius one for the Frobenius extension A/B. In particular, the map h corresponding to the identity 1 under the B-A-isomorphism $A \simeq \mathrm{Hom}(A_B, B_B)$ is called a *Frobenius map*.

The following, which can be seen easily, is well-known.

Lemma 1. *Let A/B be a Frobenius extension and $(h\,;\, r_i, l_i)_i$ its Frobenius system. Then*

(1) *For any ${}_AX$, a mapping $u_X : \mathrm{Hom}({}_AX, {}_AA) \to \mathrm{Hom}({}_BX, {}_BB)$ defined by $u_X(f) = h \circ f$ is an isomorphism, whose inverse is given by $u_X^{-1}(g)(x) = \sum_i r_i g(l_i x)$.*

(2) *For any Y_B, a mapping $v_Y : Y \otimes {}_BA \to \mathrm{Hom}(A_B, Y_B)$ defined by $v_Y(y \otimes a) = y \circ h \circ a$ is an isomorphism, whose inverse is given by $v_Y^{-1}(k) = \sum_i k(r_i) \otimes l_i$.*

A ring extension A/B is said to be separable if the canonical epimorphism φ: $A \otimes {}_B A \rightarrow A$ defined by $\varphi(x \otimes y) = xy$ splits as an A-A-bimodule map. A B-B-subbimodule X of A is said to be invertible if there exists a B-B-subbimodule Y of A such that $XY = YX = B$ ([6]).

The following is an extended version of Handelman and Renault [3, Theorem 1] and its proof runs after them.

Lemma 2. *Let A/B be a separable extension. Suppose either there exist elements a_i's in A such that $A = \sum_i a_i B$ and $a_i B = B a_i$ for all i or there exist invertible B-B-subbimodules X_i's of A such that $A = \sum_i X_i$. If B is a finite product of simple rings, then A is also a finite product of simple rings.*

Proof. Noting B semisimple as a B-B-bimodule, A is semisimple as a B-B-bimodule in either case. Since A/B is separable, we have finitely many elements $u_j, v_j \in A$ $(j = 1, \ldots, n)$ such that $\sum_{j=1}^n x u_j \otimes v_j = \sum_{j=1}^n u_j \otimes v_j x$ in $A \otimes {}_B A$ for all $x \in A$ and $\sum_{j=1}^n u_j v_j = 1$. Let I be any two-sided ideal of A. Let $f: A \rightarrow I$ be a B-B-bimodule map fixing I pointwise. Then one can see a mapping $F: A \rightarrow I$ defined by $F(x) = \sum_{j,k=1}^n (u_j f(v_j(x u_k))) v_k$ an A-A-bimodule map fixing I pointwise. Hence A is semisimple as an A-A-bimodule, and so it is a finite product of simple rings.

Let G be a group. A ring A is called a G-graded ring if we can write A as a direct sum of additive subgroups $A = \bigoplus_{\sigma \in G} A(\sigma)$ where $A(\sigma) A(\tau) \subset A(\sigma\tau)$ for all $\sigma, \tau \in G$. A is said to be strongly G-graded if $A(\sigma) A(\tau) = A(\sigma\tau)$ for all $\sigma, \tau \in G$.

The following is due to Miyashita [6, Theorem 2.11].

Lemma 3. *Let $A = \bigoplus_{\sigma \in G} A(\sigma)$ be a strongly G-graded ring with G a finite group. Then A is a Frobenius extension of $A(1)$. If, in addition, the order of G is invertible in $A(1)$ then A is a separable extension of $A(1)$.*

From Lemmas 2 and 3, we have

Proposition 4. *Let $A = \bigoplus_{\sigma \in G} A(\sigma)$ be a strongly G-graded ring. Suppose G is a finite group and its order is invertible in $A(1)$. If $A(1)$ is a finite product of simple rings then A is also a finite product of simple rings.*

Recall that a ring A is biregular if for any $a \in A$, the two-sided ideal AaA is generated by a central idempotent. It is well-known that any biregular ring is non-singular on both sides, hence a biregular right self-injective ring is also regular in the sense of von Neumann.

The next is a generalization of [3, Corollary 2] to strongly G-graded rings of finite groups. The proof is essentially same as in there.

Corollary. *Under the same notation and assumption as in Proposition 4, if $A(1)$ is a biregular right self-injective ring, then A is also a biregular right self-injective ring.*

Proof. Since $A(1)$ is a regular right self-injective ring, A is so from Lemma 3. By [9, Proposition 1.6], it suffices to show that every prime ideal of A is maximal. Let P be a prime ideal of A and let $B = A(1)$. Then, by [7, Theorem 4.1], we have $P \cap B = \bigcap_{\sigma\in G} Q^{\sigma}$ for some prime ideal Q of B, where $Q^{\sigma} = A(\sigma^{-1})\, QA(\sigma)$. But Q is maximal by our assumption on B, hence the factor ring $\bar{B} = B/B \cap P$ is a finite product of simple rings. Since the factor ring $\bar{A} = A/P$ is a strongly G-graded ring with $\bar{A}(1)$ isomorphic to $\bar{B}$, $\bar{A}$ is a finite product of simple rings by Proposition 4, which implies P maximal.

Let G be a finite group acting as automorphisms on A. Let $\Delta = \Delta(A\,;G)$ be the trivial crossed product of A relative to G. Δ is a free (left) A-module with free generator $\{u_\sigma\}$ indexed by G. It becomes a ring by means of $xu_\sigma yu_\tau = x\sigma(y)u_{\sigma\tau}$ and has u_1 for its identity. Moreover, it is a G-graded ring and a mapping $x \to xu_1$ imbeds A as a subring of Δ. Further, A can be regarded as a left Δ-module by means of $(au_\sigma)*x = a\sigma(x)$, and the endomorphism ring of the left Δ-module A is anti-isomorphic to the fixed subring A^G. Let $j : \Delta \to \mathrm{BiEnd}({}_\Delta A)$ be the canonical ring homomorphism given by $j(\sum_\sigma a_\sigma u_\sigma)(x) = \sum_\sigma a_\sigma \sigma(x)$, where $\mathrm{BiEnd}({}_\Delta A)$ denotes the biendomorphism ring of ${}_\Delta A$.

We are now in position to state the main theorem of this paper.

Theorem 5. *Let G be a finite group acting as automorphisms on A, Δ the trivial crossed product of A relative to G and $B = A^G$. Let I, J be the trace ideal and the annihilator of the left Δ-module A, respectively; $I = \sum\{\mathrm{Im}(f)\,;\ f: {}_\Delta A \to {}_\Delta\Delta\}$, $J = \{d \in \Delta\,;\ d*x = 0$ for all $x \in A\}$. Then the following statements (1) and (2) are equivalent.*

(1) i) *A is finitely generated projective as a right B-module.*

ii) *$\mathrm{Hom}(A_B, B_B)$ is generated by the trace $tr : A \to B$ as a right A-module.*

(2) $\Delta = I + J$.

Moreover, A is a Frobenius extension of B having the trace $\mathrm{tr}: A \to B$ *as a Frobenius map if and only if* $\Delta = I \oplus J$.

Proof. Recall that a mapping $h: \Delta \to A$ defined by $h(\sum_\sigma a_\sigma u_\sigma) = a_1$ gives a Frobenius system $(h\,;\ u_\sigma, u_\sigma^{-1})_{\sigma \in G}$ of the ring extension Δ/A.

Assume (2) Let $1 = d + d'$ $(d \in I,\ d' \in J)$ and $d = \sum_{i=1}^{m} f_i(x_i)$ for $f_i: {}_\Delta A \to {}_\Delta \Delta$, $x_i \in A$. Applying Lemma 1 to the Frobenius extension Δ/A with the above system, the canonical homomorphism $u_A: \mathrm{Hom}({}_\Delta A, {}_\Delta \Delta) \to \mathrm{Hom}({}_A A, {}_A A)$ given in there is isomorphic. Setting $y_i = u_A(f_i)(1) \in A$ $(i = 1, \ldots, m)$, we have

$$(*) \qquad d = \sum_{\sigma \in G} \left(\sum_{i=1}^{m} x_i \sigma(y_i)\right) u_\sigma$$

$$(**) \qquad x = \sum_{i=1}^{m} x_i \,\mathrm{tr}(y_i x) \text{ for all } x \in A,$$

which yields (1).

Conversely assume the conditions i) and ii). Let x_i's and y_i's $(i = 1, \ldots, m)$ be elements of A satisfying the condition $(**)$, and let d the element of Δ given by $(*)$. We have then $d - 1 \in J$, and so $\Delta = I + J$. Assume further $I \cap J = 0$, and let $\mathrm{tr} \circ a = 0$ for a in A. Since $\sum_{\sigma \in G} \sigma(a) u_\sigma = u_A^{-1}(a^r)(1) \in I \cap J$, we have $a = 0$. Here a^r denotes the right multiplication induced by a; $a^r: A \to A$, $x \to xa$. It follows that A is a Frobenius extension of B having $\mathrm{tr}: A \to B$ as a Frobenius map.

Assume now that A/B is such a Frobenius extension. Let d be an arbitrary element of the intersection of I and J. Since d is contained in I, there exist a finite number of elements x_i's and y_i's in A satisfying $(*)$ as mentioned above. But $d * x = \sum_i x_i \,\mathrm{tr}(y_i x) = 0$ for all x in A, and hence $\sum_i x_i \otimes y_i = 0$ in $A \otimes_B A$ by Lemma 1(2). It follows that $\sum_i x_i \sigma(y_i) = 0$ for all σ in G, and so $d = 0$, proving the theorem.

Corollary. *If A is a Frobenius extension of A^G having the trace* $\mathrm{tr}: A \to A^G$ *as a Frobenius map, then* $j: \Delta \to \mathrm{BiEnd}({}_\Delta A)$ *is an epimorphism.*

As an application of the preceding theorem, we have

Proposition 6. *Let G be a finite group acting as automorphisms on A. Assume that the trivial crossed product Δ of A relative to G is biregular and that* $\mathrm{tr}(c) = 1$ *for some $c \in A$. Then A is a Frobenius extension of A^G having* $\mathrm{tr}: A \to A^G$ *as a Frobenius map.*

Proof. Let I, J be same as in Theorem 5. Let $e = \sum_{\sigma \in G} \sigma(c) u_\sigma \in \Delta$. Then $e = e^2$ and $A \simeq \Delta e$ as left Δ-modules, and so $I = \Delta e \Delta$. But, since Δ

is biregular, $I=\Delta f$ for some central idempotent f of Δ. It follows that $\Delta = I \oplus J$. Thus the proposition follows from Theorem 5.

As a consequence of Proposition 6, we have the following which sharpens slightly Theorem 4 of [3] and extends a result of [5].

Corollary 1. *Let G be a finite group acting as automorphisms on A. Assume that the order of G is invertible in A. If either A is a finite product of simple rings or A is a biregular right self-injective ring, then A is a Frobenius extension of A^G having* $\mathrm{tr}: A \to A^G$ *as a Frobenius map.*

Proof. In either case, Δ is biregular from Proposition 4 and its corollary, and so this follows from Proposition 6.

Corollary 2. *Let G be a finite group acting as automorphisms on A. Assume that the order of G is invertible in A. If A is a commutative, von Neumann regular ring, then A is a Frobenius extension of A^G having* $\mathrm{tr}: A \to A^G$ *as a Frobenius map, and moreover A is separable over A^G.*

Proof. Let $|G|$ be the order of G. Set $e = |G|^{-1} \sum_{\sigma \in G} u_\sigma \in \Delta$ and $B = A^G$. Then $e = e^2$ and $A \simeq \Delta e$ as left Δ-modules. Since Δ is a von Neumann regular ring, $e\Delta e$, and hence B is also a von Neumann regular ring. Let I and J be same as in Theorem 5, and put $K = I + J$. Then $I = \Delta e \Delta$ and $J = l_\Delta(\Delta e)$, the left annihilator of Δe in Δ. Obviously, $I \cap J = 0$. We shall show $K = \Delta$. Let $\mathfrak{m}$ be an arbitrary maximal ideal of B. Then the factor rings $A/\mathfrak{m}A$ and $B/\mathfrak{m}$ are isomorphic to the localizations $A_\mathfrak{m}$ and $B_\mathfrak{m}$ of A and B at $\mathfrak{m}$, respectively, and $(A_\mathfrak{m})^G = B_\mathfrak{m}$. Thus $A_\mathfrak{m}$ is finitely generated over $B_\mathfrak{m}$ by [2, Theorem 3]. It follows from Corollary 1 and Theorem 5 that $K_\mathfrak{m} = \Delta(A_\mathfrak{m}; G)e\Delta(A_\mathfrak{m}; G) \oplus l_{\Delta(A_\mathfrak{m}; G)}(\Delta(A_\mathfrak{m}; G)e) = \Delta(A_\mathfrak{m}; G) = \Delta_\mathfrak{m}$, which implies that $K = \Delta$ as desired. Hence, by Theorem 5, A is a Frobenius extension of B having $\mathrm{tr}: A \to B$ as a Frobenius map. Noting $|G|^{-1} \in A$, A is separable over B from [1, Proposition A.4].

Remark. Combining the above result with [9, Corollary 1.2], Δ is biregular if A is commutative, von Neumann regular and $|G|^{-1} \in A$.

The following is an extended version of Corollary 1.

Proposition 7. *Let G, Δ, I and J be same as in Theorem 5. Let N be the Jacobson radical of A, and let K be the set consisting of $d \in \Delta$ with $d * x \in$*

N for all $x \in A$. Assume that $|G|^{-1} \in A$ and that either the factor ring $\bar{A} = A/N$ is a finite direct product of simple rings or $\bar{A}$ is a biregular right self-injective ring. Then the following are equivalent.

(1) *A is a Frobenius extension of A^G having $\mathrm{tr} : A \to A^G$ as a Frobenius map.*

(2) *$I \cap J = 0$ and $K = J + \sum_{\sigma \in G} Nu_\sigma$.*

Proof Let $e \in \Delta$ be same as in the proof of Corollary 2. Then $A \simeq \Delta e$ as left Δ-modules and $I = \Delta e \Delta$. Since G acts as automorphisms on $\bar{A}$ in a natural way, we have a mapping $f: \Delta \to \Delta(\bar{A}; G)$ defined by $f(\sum_\sigma a_\sigma u_\sigma) = \sum_\sigma \bar{a}_\sigma u_\sigma$. Then $\mathrm{Ker}(f) = \sum_\sigma Nu_\sigma$ and $f(K)$ is the left annihilator of $\bar{A}$ in $\Delta(\bar{A}; G)$. By Corollary 1 and Theorem 5, we have $f(\Delta) = f(\Delta)f(e)f(\Delta) \oplus f(K)$, which yields that $\Delta = \Delta e \Delta + K$ and $\Delta e \Delta \cap K = \mathrm{Ker}(f)$.

(1) $\Rightarrow$ (2) By Theorem 5, $\Delta = I + J$ and $I \cap J = 0$. Let $L = J + \sum_\sigma Nu_\sigma$. Obviously, L is contained in K. But, noting $I \cap K = \mathrm{Ker}(f)$, L contains K, hence $L = K$.

(2) $\Rightarrow$ (1) Since $\Delta = \Delta e \Delta + K$, we have $\Delta = (\Delta e \Delta + J) + \sum_\sigma Nu_\sigma = (\Delta e \Delta + J) + \mathrm{Rad}({}_A\Delta)$, where $\mathrm{Rad}({}_A\Delta)$ denotes the radical of the left A-module Δ. It follows that $\Delta = \Delta e \Delta + J$, which yields (1) from Theorem 5.

REFERENCES

[1] M. AUSLANDER and O. GOLDMAN : The Brauer group of a commutative ring, Trans. Amer. Math. Soc. **97** (1960), 367–409.

[2] D. FARKAS and R. SNIDER : Noetherian fixed rings, Pacific J. Math. **69** (1977), 347–353.

[3] D. HANDELMAN and G. RENAULT : Actions of finite groups on self-injective rings, Pacific J. Math. **89** (1980), 69–80.

[4] F. KASCH : Projective Frobenius-Erweiterungen, Sitzungsber. Heidelberger Akad. Wiss. 1960/61 (4), 89–109.

[5] Y. KITAMURA : Fixed rings of simple rings, Math. J. Okayama Univ. **30** (1988), 21–24.

[6] Y. MIYASHITA : On Galois extensions and crossed products, J. Fac. Sci. Hokkaido Univ. **21** (1970), 97–121

[7] C. NASTASESCU : Strongly graded rings of finite groups, Comm. Algebra **11** (1983), 1033–1071.

[8] T. ONODERA : Some studies on projective Frobenius extensions, J. Fac. Sci. Hokkaido Univ. **18** (1964), 89–107.

[9] G. RENAULT : Anneaux bireguliers auto-injectifs a droite, J. Algebra **36** (1975), 77–84.

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