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ON GALOIS H-DIMODULE ALGEBRAS

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Let k be a commutative ring and H a finite commutative, cocommutative Hopf k-algebra with dual $H^* = \mathrm{Hom}_k(H, k)$. An H-Galois extension A of k is called a Galois H-dimodule algebra, [1], [4], [5], if A has an additional structure of an H-module (algebra) satisfying

$$h(ga) = g(ha), \quad a \in A,\ h \in H,\ g \in H^*.$$

By definition, a morphism of Galois H-dimodule algebras is an H-linear morphism of the underlying Galois extensions. It is shown in this paper that the set $D(H)$ of isomorphism classes of Galois H-dimodule algebras decomposes into

$$D(H) = E(H) \times \mathrm{Hopf}_k(H, H^*) \tag{1}$$

where $E(H)$ is the group of isomorphism classes of H-Galois extensions of k, and $\mathrm{Hopf}_k(H, H^*)$ the group of Hopf algebra homomorphisms $H \to H^*$. In [1] a certain product for Galois H-dimodule algebras was defined, and [1], prop. 2.4, says that $D(H)$ is a group. This generalizes [5], theorem 1.9, which states that $D(H)$ is a monoid if H is a free k-module. Under (1) the product reads

$$(A, \varphi).(B, \psi) = (A^{\varphi}. B, \varphi\psi) \tag{2}$$

where A^{φ} equals A as H-comodule and has multiplication

$$a.b = \sum a_{(0)}(\varphi(a_{(1)})b), \quad a, b \in A.$$

But if the product (2) is associative, or has a unit element, then $A^{\varphi} \cong A$ for all A and φ, and we shall give examples where $H^{\varphi} \not\cong H$. Consequently, we obtain that the above mentioned results are not correct, even in the special cases considered in [5], theorems 2.4 and 2.6.

Notations. The counit of H(or H^*) will be denoted by ε, its antipode by λ. For H-Galois extensions A and B of k their product $A.B$ is defined to be the equalizer of the comodule structures

$$A.B \to A \otimes B \rightrightarrows A \otimes B \otimes H,$$

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where $\otimes = \otimes_k$. We refer to [2] for a proof that $A.B$ agrees with various other definitions of the product of Galois extensions.

In the following let A be an H-Galois extension of k. Thus A is an H-comodule algebra and the map

$$\omega\colon H^* \otimes A \to \mathrm{End}_k(A), \quad \omega(g \otimes b)(a) = (ga)b,$$

is an isomorphism where $ga = \sum a_{(0)} \langle g, a_{(1)} \rangle$, $g \in H^*$, $a, b \in A$, (see for example [8]). Also, ω induces an isomorphism

$$(3) \qquad H^* \xrightarrow{\sim} \mathrm{End}_{H^*}(A).$$

Indeed, consider $w \in H^* \otimes A$ such that $\omega(w)$ is H^*-linear. Then since $g(ab) = \sum g_{(1)}a . g_{(2)}b$ for all $g \in H^*$, and $a, b \in A$, we have

$$w(g \otimes 1) = (\sum g_{(1)} \otimes g_{(2)}) w, \quad g \in H^*.$$

Therefore, $\varepsilon(g)w = (\sum \lambda(g_{(1)})g_{(2)} \otimes 1)w = \sum (\lambda(g_{(1)}) \otimes 1)w(g_{(2)} \otimes 1) = \sum (\lambda(g_{(1)})g_{(2)} \otimes g_{(3)})w = (1 \otimes g)w$. Hence $w \in H^* \otimes A^{H^*} \cong H^*$. Since A is finitely generated and projective over H^*, (3) says that A is an invertible H^*-module.

Assume in the following that A is also an H-module satisfying $h(ga) = g(ha)$ for all $h \in H$, $g \in H^*$, and $a \in A$. By (3) this means we are given a k-algebra homomorphism $\varphi\colon H \to H^*$ such that

$$(4) \qquad ha = \varphi(h)a = \sum a_{(0)} \langle \varphi(h), a_{(1)} \rangle, \quad a \in A, h \in H.$$

Proposition 1. *A is an H-module algebra with respect to* (4) *if and only if φ is a homomorphism of Hopf k-algebras.*

Proof. That A be an H-module algebra means

$$(5) \qquad \varphi(h)(ab) = \sum \varphi(h_{(1)})a . \varphi(h_{(2)})b, \quad h \in H, a, b \in A,$$

and $\varepsilon(h)1_A = \varphi(h)1_A = \varepsilon\varphi(h)1_A$. Since $\varphi(h)(ab) = \sum \varphi(h)_{(1)}a . \varphi(h)_{(2)}b$, A is an H-module algebra if φ is a coalgebra morphism. Conversely, let $x, y \in H$ and choose $\sum_i a_i \otimes b_i$ and $\sum_j c_j \otimes d_j$ in $A \otimes A$ such that

$$(6) \qquad 1 \otimes x = \sum a_i b_{i(0)} \otimes b_{i(1)}$$

and $1 \otimes y = \sum c_{j(0)} d_j \otimes c_{j(1)}$ holds in $A \otimes H$. Then assuming (5) we have

$$\begin{aligned}\langle \varphi(h), xy\rangle &= \sum a_i b_{i(0)} c_{j(0)} d_j \langle \varphi(h),\ b_{i(1)} c_{j(1)}\rangle \\ &= \sum a_i(\varphi(h)(b_i c_j))d_j \\ &= \sum a_i(\varphi(h_{(1)})b_i)(\varphi(h_{(2)})c_j)d_j \\ &= \sum \langle \varphi(h_{(1)}),\ x\rangle\ \langle \varphi(h_{(2)}),\ y\rangle.\end{aligned}$$

Since $\langle \varphi(h), xy\rangle = \sum \langle \varphi(h)_{(1)}, x\rangle\ \langle \varphi(h)_{(2)}, y\rangle$ this implies that φ is a coalgebra morphism as required.

Given $\varphi \in \mathrm{Hopf}_k(H, H^*)$ we define on A a new multiplication by

$$a.b = \sum a_{(0)} b_{(0)} \langle \varphi(a_{(1)}),\ b_{(1)}\rangle, \quad a,\ b \in A.$$

This gives us an H-comodule algebra A^{φ} having the same comodule structure as A.

Lemma 2. *A^{φ} is an H-Galois extension of k.*

Proof. Let $u \in H^* \otimes H^*$ denote the preimage of φ under the canonical isomorphism $H^* \otimes H^* \xrightarrow{\sim} \mathrm{Hom}_k(H, H^*)$, $(g \otimes g')(h) = \langle g, h\rangle g'$. Then u is a 2-cocycle, (cf. [6], sec. 1.5), and if $A = H$ then A^{φ} coincides with the (associative) H-Galois algebra $(H, \bar{u})$ of [3], prop. 1.6. But

$$A^{\varphi} \xrightarrow{\ \sim\ } A.H^{\varphi} \subset A \otimes H,\ a \to \sum a_{(0)} \otimes a_{(1)},$$

is an isomorphism of H-comodule algebras, and this completes the proof.

Obviously $A^{\varepsilon} = A$ and since the product in $\mathrm{Hopf}_k(H, H^*)$ is defined by $(\varphi\psi)(h) = \sum \varphi(h_{(1)})\,\psi(h_{(2)})$, $h \in H$, one has

(7) $$(A^{\varphi})^{\psi} = A^{\varphi\psi}.$$

Let $D(H)$ denote the set of isomorphism classes of Galois H-dimodule algebras.

Theorem 3. *By viewing each Galois H-dimodule algebra as a pair (A, φ) as above one obtains a bijection*

(8) $$D(H) \xrightarrow{\sim} E(H) \times \mathrm{Hopf}_k(H, H^*).$$

It transforms the product of Galois H-dimodule algebras of [1] into

(9) $$(A, \varphi).(B, \psi) = (A^{\varphi}.B, \varphi\psi)$$

where $A^{\varphi}.B$ denotes the product of the Galois extensions A^{φ} and B.

Proof. Let $f : (A, \varphi) \to (B, \psi)$ be a homomorphism of Galois H-dimodule algebras; to prove (8) it suffices to show $\varphi = \psi$. Let $x \in H$ and write $1 \otimes x$ in $A \otimes H$ as in (6). So $1 \otimes x = \sum f(a_i)f(b_{i(0)}) \otimes b_{i(1)}$. Since $f(\varphi(h)b_i) = \psi(h)f(b_i)$ for $h \in H$ we obtain

$$\begin{aligned}\langle \varphi(h), x\rangle &= \sum f(a_i)f(b_{i(0)})\langle \varphi(h), b_{i(1)}\rangle \\ &= \sum f(a_i)f(b_{i(0)})\langle \psi(h), b_{i(1)}\rangle = \langle \psi(h), x\rangle.\end{aligned}$$

Thus $\varphi(h) = \psi(h)$ and we are done. Next consider the product $\mathbf{A}.\mathbf{B}$ of any Galois H-dimodule algebras $\mathbf{A} = (A, \varphi)$ and $\mathbf{B} = (B, \psi)$. By definition, [1], 2.1, we have $\mathbf{A}.\mathbf{B} = (A^{\varphi}.B, \varphi\psi)$ as H-comodules. Furthermore, also the H-module structures are the same. For let $h \in H$ and $\sum_i a_i \otimes b_i \in \mathbf{A}.\mathbf{B}$; by abuse of notation we denote the latter by $a \otimes b$. Then

$$\begin{aligned}h.(a \otimes b) &= \sum h_{(1)}.a \otimes h_{(2)}.b \\ &= \sum a_{(0)}\langle \varphi(h_{(1)}), a_{(1)}\rangle \otimes b_{(0)}\langle \psi(h_{(2)}), b_{(1)}\rangle \\ &= \sum a_{(0)}\langle \varphi(h_{(1)}), a_{(1)}\rangle \otimes \langle \psi(h_{(2)}), a_{(2)}\rangle b \\ &= \sum a_{(0)}\langle \varphi(h_{(1)})\psi(h_{(2)}), a_{(1)}\rangle \otimes b \\ &= (\varphi\psi)(h)(a \otimes b).\end{aligned}$$

Finally, multiplication in $\mathbf{A}.\mathbf{B}$ is defined by the smash product formula. This means

$$\begin{aligned}(a \otimes b).(c \otimes d) &= \sum a(b_{(1)}c) \otimes b_{(0)}d \\ &= \sum ac_{(0)}\langle \varphi(b_{(1)}), c_{(1)}\rangle \otimes b_{(0)}d \\ &= \sum a_{(0)}c_{(0)}\langle \varphi(a_{(1)}), c_{(1)}\rangle \otimes bd\end{aligned}$$

since $\sum a \otimes b_{(0)} \otimes b_{(1)} = \sum a_{(0)} \otimes b \otimes a_{(1)}$. Hence $\mathbf{A}.\mathbf{B} = A^{\varphi}.B$ as k-algebras and this completes the proof.

Lemma 4. *If the product* (9) *is associative, or has a unit element, then* $A^{\varphi} \cong A$ *(as Galois extensions) for all* $\varphi \in \mathrm{Hopf}_k(H, H^*)$ *and H-Galois extensions A of k.*

Proof. Associativity of (9) implies therefore $(A^{\varphi}B)^{\varphi\psi} \cong A^{\varphi}B^{\psi}$ because $E(H)$ is a group. Since $(\lambda\varphi).\varphi = \varepsilon$ holds in $\mathrm{Hopf}_k(H, H^*)$, we have $(H^{\lambda\varphi})^{\varphi} = H^{\varepsilon} = H$. Setting $A = H^{\lambda\varphi}$ we obtain $B^{\varphi\psi} \cong B^{\psi}$ by (7). Hence $B^{\varphi} \cong B$. The other statement is evident.

Remark 5. Given H-dimodule algebras $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$, the natural map

A # (B # C) $\to$ (A # B) # C is an isomorphism of H-dimodule algebras by [4], thm. 3.3. But in general the inclusion B.C $\to$ B # C is not H-colinear, because the comodule structure of B # C is given by $b \otimes c \to \sum b_{(0)} \otimes c_{(0)} \otimes b_{(1)}c_{(1)}$, whereas that of B.C by $b \otimes c \to \sum b_{(0)} \otimes c \otimes b_{(1)} = \sum b \otimes c_{(0)} \otimes c_{(1)}$. It follows that, in general, A.(B.C) is not a subalgebra of A # (B # C).

In the following, let $\bar{\mathrm{A}}'$ for $\mathrm{A} = (A, \varphi)$ be defined as in [1], and let A^{-1} be the H-Galois extension A^{op} having H-comodule structure $a \to \sum a_{(0)} \otimes \lambda(a_{(1)})$. We claim that $\bar{\mathrm{A}}' = ((A^{-1})^{\varphi}, \lambda\varphi)$. It is clear that both sides have the same H-comodule structure, and that they are equal to A as H-modules. Furthermore, by [1], prop. 1.2c), multiplication of $\bar{\mathrm{A}}'$ is given by $a.b = \sum b_{(0)}a_{(0)}\langle \varphi(a_{(1)}), b_{(1)}\rangle$ since $\lambda^2 = \mathrm{id}$. However, $\langle \varphi(\alpha), \beta\rangle = \langle \varphi(\alpha), \lambda^2(\beta)\rangle = \langle \lambda\varphi(\alpha), \lambda(\beta)\rangle = \langle \varphi(\lambda(\alpha)), \lambda(\beta)\rangle$ for $\alpha, \beta \in H$. Hence the multiplication of $\bar{\mathrm{A}}'$ coincides with that of $(A^{-1})^{\varphi}$, and this proves our claim. Obviously, one has $\mathrm{A}.\bar{\mathrm{A}}' = A.A^{-1}$ as H-comodules but, in general, not as k-algebras. Consider then the isomorphism

$$(10) \qquad A \otimes A \xrightarrow{\sim} A \otimes H, \ a \otimes b \to \sum ab_{(0)} \otimes \lambda(b_{(1)}).$$

It is shown in the proof of [1], prop. 2.4, that restricting (10) gives an isomorphism of H-comodules

$$f: \ \mathrm{A}.\bar{\mathrm{A}}' \xrightarrow{\sim} k \otimes H = H.$$

Furthermore, it is claimed there that f is also a morphism of k-algebras. But in general this fails to be true. Actually, f respects multiplication of $A.A^{-1}$. Indeed, let (by abuse of notation) $a \otimes b$, $c \otimes d \in A.A^{-1}$ and write $f(c \otimes d) = 1 \otimes h$. Then

$$\begin{aligned} f(ac \otimes db) &= \sum acd_{(0)}b_{(0)} \otimes \lambda(b_{(1)})\lambda(d_{(1)}) \\ &= \sum (a \otimes \lambda(b_{(1)}))(cd_{(0)} \otimes \lambda(d_{(1)}))(b_{(0)} \otimes 1) \\ &= \sum (a \otimes \lambda(b_{(1)}))(b_{(0)} \otimes h) = f(a \otimes b)f(c \otimes d). \end{aligned}$$

In other words, $f((a \otimes b^{\mathrm{op}})(c \otimes d^{\mathrm{op}})) = f(a \otimes b^{\mathrm{op}})f(c \otimes d^{\mathrm{op}})$, so that $f: \ A.A^{-1} \to H$ is an algebra homomorphism. This shows that A^{-1} represents the inverse of the class of A in the abelian group $E(H)$. (Actually the H-Galois extensions of k form a symmetric Picard category in the sense of [9]).

Remark 6. The Hopf k-algebra homomorphism $\varphi: \ H \to H^*$ is bijective if and only if (A, φ) is an H^*-Galois extension of k. For example this is the case for Kummer extensions of k, cf. [7].

Let $H=k[G]$, G a finite abelian group. Then one has a natural isomorphism of $\mathrm{Hopf}_k(H, H^*)$ with the group of bimultiplicative mappings of $G\times G$ into the group k^* of units of k. Then $k[G]^{\varphi}\cong k[G]$ if and only if $\varphi: G\times G\to k^*$ is a 2-coboundary, and in general there are pairings φ which define non-trivial cocycles, even if G has order two, cf. [10], p. 160, remark.

Finally let k be a field of characteristic 2, and let $H=k[X]/X^2=k[\delta]$ be as in [5], 2.2. Let $(1, \delta^*)$ be the dual basis of $(1, \delta)$. Then

$$\eta: k^+\to \mathrm{Hopf}_k(H, H^*),\ \eta(t)(\delta)=t\delta^*,\quad t\in k,$$

is an isomorphism of groups. Further, one has $H^{\eta(t)}\cong k[X]/(X^2-t)$. So $H\cong H^{\eta(t)}$ implies that t is a square in k.

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