Math. J. Okayama Univ. 31 (1989), 73–77

ON COMMUTATIVE RINGS OVER WHICH ALL SEMIPRIME FINITELY GENERATED ALGEBRAS ARE SEPARABLE

Dedicated to Professor Takasi Nagahara on his 60th birthday

YASUYUKI HIRANO and JAE KEOL PARK

In [1], Armendariz proved that every semiprime finitely generated algebra over a commutative von Neumann regular ring is an Azumaya algebra over its center. In this paper, we shall first prove the converse, that is, if every semiprime finitely generated algebra over a commutative semiprime ring R is Azumaya (over its center), then R is von Neumann regular. Using this, we shall characterize a commutative quasi-regular ring in terms of Azumaya algebra. Finally, we shall describe the structure of a commutative ring over which every semiprime finitely generated algebra is separable.

Throughout we will assume that rings have unit. Let R be a commutative ring. We describe an R-algebra A as finitely generated or faithful if A is finitely generated or faithful when considered as an R-module. The Jacobson radical of a ring A will be denoted by $J(A)$, the prime radical by $P(A)$, and the center by $Z(A)$.

An element a of a ring R is called *von Neumann regular* if there exists an $x \in R$ such that $a = axa$. We start with the following

Lemma 1. *Let A be a semiprime ring with center R and let a be an element of R. Then $B = \begin{pmatrix} A & aA \\ aA & A \end{pmatrix}$ is Azumaya if and only if A is Azumaya and a is von Neumann regular in R.*

Proof. Suppose that B is Azumaya. It is easily checked that the center of B is $Z(B) = \left\{ \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \middle| x, y \in R, (x-y)a = 0 \right\}$. Clearly, $I = \begin{pmatrix} aA & aA \\ aA & aA \end{pmatrix}$ is an ideal of B. Since R is semiprime, we obtain $I \cap Z(B) = \left\{ \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix} \middle| x \in aR \right\}$. Hence, by [3, Corollary 2.3.7], we have $\begin{pmatrix} aA & aA \\ aA & aA \end{pmatrix} = \begin{pmatrix} aA & a^2A \\ a^2A & aA \end{pmatrix}$. Hence there exists $x \in A$ such that $a^2x = a$. If we set $z = ax^2$, then we can easily check that $z \in R$ and $a^2z = a$. Therefore a is von Neumann regular in R. Clearly, $e = az$ is a central idempotent of A, and so we have $B \cong M_2(eA) \oplus (1-e)A \oplus (1-e)A$. Hence A is Azumaya (cf. [3, Proposition 2.1.13]).

Now the reverse implication is clear.

We come to our first theorem.

Theorem 1. *Let R be a commutative semiprime ring. Then the following statements are equivalent :*

(1) *R is von Neumann regular.*

(2) *Every semiprime finitely generated R-algebra is Azumaya (over its center).*

(3) *For every finitely generated R-algebra A, $J(A)$ is nilpotent and $A/J(A)$ is Azumaya.*

Proof. (1) ⇒ (2). This follows from [1, Theorem 2].

(2) ⇒ (1). Take an arbirary $a \in R$, and consider the R-algebra $A = \begin{pmatrix} R & aR \\ aR & R \end{pmatrix}$. Then A is a semiprime finitely generated R-algebra. Then A is Azumaya, and hence a is von Neumann regular in R by Lemma 1.

(2) ⇒ (3). By the equivalence of (1) and (2), R is von Neumann regular. Hence this follows from [6, Proposition 2.2].

(3) ⇒ (2). Let A be a semiprime finitely generated R-algebra. Then, since $J(A)$ is nilpotent, the semiprimeness implies $J(A) = 0$. Hence $A(= A/J(A))$ is Azumaya.

Let A and B be rings with the same identity such that $A \subseteq B$. Then B is called a *finite liberal extension* of A if it contains a finite set of A-centralizing elements, $\{a_1, \dots, a_n\}$ say, such that $B = Aa_1 + \cdots + Aa_n$. Now we deal with the noncommutative version of Theorem 1.

Theorem 2. *Let A be a semiprime ring with center R. Then the following statements are equivalent :*

(1) *Every semiprime finite liberal extension of A is Azumaya.*

(2) *A is a finitely generated R-algebra and R is von Neumann regular.*

Proof. (1) ⇒ (2). Since A is a semiprime finite liberal extension of itself, A is Azumaya. Hence A is finitely generated over its center R. Take an arbitrary $a \in R$, and consider the ring $B = \begin{pmatrix} A & aA \\ aA & A \end{pmatrix}$. It is easily checked that B is a semiprime finite liberal extension of A. Hence B is Azumaya, and so a is von Neumann regular in R by Lemma 1.

(2) $\Rightarrow$ (1). If B is a semiprime finite liberal extension of A, then B is a finitely generated R-algebra, and so B is Azumaya by Theorem 1.

Let R be a commutative ring and let S be the set of non-zero-divisors of R. Then R is said to be *quasi-regular* provided its total quotient ring $S^{-1}R$ is von Neumann regular ([4]). For example, p.p. rings are quasi-regular (see e.g. [5]).

Theorem 3. *Let R be a commutative semiprime ring with total quotient ring Q. Then the following statements are equivalent :*

(1) *R is quasi-regular.*

(2) *For every semiprime finitely generated faithful R-algebra A, $A \otimes_R Q$ is Azumaya.*

(3) *For every semiprime finitely generated faithful R-algebra A, there exists a non-zero-divisor $d \in R$ such that $A_d = A \otimes_R R[d^{-1}]$ is Azumaya.*

Proof. (1) $\Rightarrow$ (2). Let A be a semiprime finitely generated faithful R-algebra. Then $A \otimes_R Q$ is a semiprime finitely generated algebra over the regular ring Q, and hence $A \otimes_R Q$ is Azumaya by Theorem 1.

(2) $\Rightarrow$ (3). Let A be a semiprime finitely generated faithful R-algebra and let S denote the set of non-zero-divisors of R. Then $S^{-1}A = A \otimes_R Q$ is Azumaya, and $Z(S^{-1}A) = S^{-1}Z(A)$. Since $S^{-1}A \otimes_{Z(S^{-1}A)} (S^{-1}A)^{op} = (A \otimes_{Z(A)} A^{op}) \otimes_{Z(A)} S^{-1}Z(A)$, a separability idempotent e for $S^{-1}A$ can be written as fd^{-1} where $f \in A \otimes_{Z(A)} A^{op}$ and $d \in S$. Then $e = fd^{-1}$ is in $A_d \otimes_{Z(A_d)} (A_d)^{op}$, and so e is a separability idempotent for the $Z(A_d)$-algebra A_d. This implies that A_d is Azumaya.

(3) $\Rightarrow$ (1). Let a be an arbitrary element of R. Then $A = \begin{pmatrix} R & aR \\ aR & R \end{pmatrix}$ is a semiprime finitely generated faithful R-algebra. By hypothesis, there exists a non-zero-divisor $d \in R$ such that $A_d = \begin{pmatrix} R[d^{-1}] & aR[d^{-1}] \\ aR[d^{-1}] & R[d^{-1}] \end{pmatrix}$ is Azumaya. By Lemma 1, a is von Neumann regular in $R[d^{-1}]$ and hence in Q. Hence every element of R is von Neumann regular in Q. Let as^{-1} be an arbitrary element of Q, where a, $s \in R$ and s is a non-zero-divisor. Then there exists $x \in Q$ such that $a^2x = a$, and so we have $(as^{-1})^2xs = as^{-1}$. Therefore Q is von Neumann regular.

Finally, we describe the structure of a commutative semiprime ring R

such that every semiprime finitely generated R-algebra is separable. To do it, we introduce the following

Definition. A commutative ring R is called *perfect von Neumann regular* if R is von Neumann regular and every prime factor ring of R is a perfect field.

Examples. (a) Every commutative von Neumann regular $\mathbf{Q}$-algebra R is perfect von Neumann regular.

(b) A ring R is called a *J-ring* if, for each $x \in R$, there exists an integer $n = n(x) > 1$ such that $x = x^n$. Clearly, J-rings are perfect von Neumann regular.

We conclude this paper with the following

Theorem 4. *Let R be a commutative semiprime ring. Then the following statements are equivalent :*

(1) *R is perfect von Neumann regular.*

(2) *Every semiprime finitely generated R-algebra is separable over R.*

(3) *Every finitely generated R-algebra contains a separable subalgebra S such that $A = S \oplus P(A)$ as R-modules.*

Proof. (1) $\Rightarrow$ (2). Let A be a semiprime finitely generated R-algebra. By [1, Theorem 2], A is Azumaya and von Neumann regular. Hence the center $Z(A)$ of A is also von Neumann regular. Since $Z(A)$ is a $Z(A)$-direct summand of A ([3, Lemma 2.3.1]), $Z(A)$ is a finitely generated R-algebra. Let M be a maximal ideal of R. Since $Z(A)/MZ(A)$ is von Neumann regular and finitely generated over R, $Z(A)/MZ(A)$ is a finite direct sum of fields each of which is a finite extension of R/M. By hypothesis, R/M is a perfect field, and hence $Z(A)/MZ(A)$ is a separable R/M-algebra. Hence, by [3, Theorem 2.7.1], $Z(A)$ is a separable R-algebra. Therefore, by [3, Theorem 2.3.8], A is a separable R-algebra.

(2) $\Rightarrow$ (3). By Theorem 1, R is von Neumann regular. Let A be a finitely generated R-algebra. Then, by [6, Proposition 2.2] we have $J(A) = P(A)$. Now the assertion follows from [2, Theorem 1].

(3) $\Rightarrow$ (2). This is trivial.

(2) $\Rightarrow$ (1). By Theorem 1, R is von Neumann regular. Hence every prime ideal of R is maximal. Let M be a prime ideal of R. Suppose that the field R/M is not perfect. Then we can find a monic polynomial $f(X) \in$

$R[X]$ such that the natural homomorphic image $\bar{f}(X) \in (R/M)[X]$ is irreducible and the field $F = (R/M)[X]/\bar{f}(X)(R/M)[X]$ is inseparable over R/M. Let A denote the R-algebra $R[X]/f(X)R[X]$. Then $B = A/P(A)$ is a semiprime finitely generated (faithful) R-algebra. However, since $B/MB(\cong F)$ is not separable over R/M, B is not a separable R-algebra by [3, Theorem 3.7.1].

Acknowledgement. The second author was supported from The Basic Science Research Center of Korea University from 1988−1989.

References

[1] E. P. Armendariz : On semiprime P.I.-algebras over commutative regular rings : Pacific J. Math. **66** (1976), 23−28.

[2] W. C. Brown : A splitting theorem for algebras over commutative von Neumann regular rings, Proc. Amer. Math. Soc. **36** (1972), 369−374.

[3] F. DeMeyer and E. C. Ingraham : Separable algebras over commutative rings, Lecture Notes in Math., vol. 181, Springer-Verlag, Berlin and New York, 1971.

[4] S. Endo : On semi-hereditary rings, J. Math. Soc. Japan **13** (1961), 109−119.

[5] M. W. Evans : On commutative p.p. rings, Pacific J. Math. **41** (1972), 687−697.

[6] J. A. Wehlen : Algebras over absolutely flat commutative rings, Trans. Amer. Math. Soc. **196** (1974), 149−160.

Department of Mathematics
Okayama University
Okayama 700, Japan
Department of Mathematics
Busan National University
Busan 609−735, Korea

(Received September 12, 1988)