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ON GENERALIZED PF-RINGS

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Throughout this paper a ring denotes a commutative ring with unity. A ring R is called a PP-ring if for every $a \in R$, the principal ideal aR is a projective R-module. Hirano [4] defined a ring R to be a generalized PP-ring (or GPP-ring) if for every $a \in R$, there exists a positive integer n such that a^nR is a projective R-module. A study of this class of rings was carried by Hirano [4]. Recall that a ring R is called a PF-ring if for every $a \in R$, the principal ideal aR is a flat R-module. Now, we define a ring R to be a generalized PF-ring (or GPF-ring) if for every $a \in R$, there exists a positive integer n such that a^nR is a flat R-module. Our aim in this paper is to study the class of GPF-rings and how it is related to GPP-rings. In § 1, we study some of the basic properties of GPF-rings. Then in § 2, we give a different proof for a result that was proved by Hirano [4].

1. Some results on GPF-ring. An ideal I of a ring R is called *pure* if for every $x \in I$, there exists $y \in I$ such that $xy = x$. It is well known that an ideal I of a ring R is pure if and only if R/I is a flat R-module, see Matlis [5]. For any $a \in R$, the mapping $f: R \to aR$ defined by $f(x) = ax$ is an R-module epimorphism. Now, the annihilator ideal, $\mathrm{ann}_R(a)$ is pure if and only $R/\mathrm{ann}_R(a)$ is a flat R-module. Since $R/\mathrm{ann}_R(a)$ is isomorphic to aR, aR is a flat R-module if and only if $\mathrm{ann}_R(a)$ is a pure ideal of R. Thus, we get the following easy lemma.

Lemma 1.1. *A ring R is a PF-ring if and only if for every $a \in R$, $\mathrm{ann}_R(a)$ is a pure ideal of R.*

Also from the above argument we obtain the following easy lemma.

Lemma 1.2. *A ring R is a GPF-ring if and only if for every $a \in R$, there exists a positive integer n such that $\mathrm{ann}_R(a^n)$ is a pure ideal of R.*

Now, we prove an easy result that will be used frequently later on.

Lemma 1.3. *Let R be a ring, and $a \in R$. If $\mathrm{ann}_R(a)$ is pure, then* *for any positive integer m,* $\operatorname{ann}_R(a^m)$ *is pure.*

Proof. Let $x \in \operatorname{ann}_R(a^m)$. Then $xa^m = 0$. If $m = 1$, we are done. If $m > 1$, then $xaa^{m-1} = 0$, and hence $xa^{m-1} \in \operatorname{ann}_R(a)$. Since $\operatorname{ann}_R(a)$ is pure, there exists $b \in \operatorname{ann}_R(a)$ such that $xa^{m-1}b = xa^{m-1}$. So, $xa^{m-1} = 0$. Inductively, we get $xa = 0$. So there exists $c \in \operatorname{ann}_R(a)$ such that $xc = x$. Since $c \in \operatorname{ann}_R(a^m)$, we are done.

Corollary 1.4. *Let R be a ring. For any* $a \in R$, *if aR is a flat R-module, then for any positive integer n,* a^nR *is a flat R-module.*

For more details about PF-rings, see Matlis [5], Al-Ezeh ([1], [2]), Al-Ezeh et al. [3] and Naoum [6].

First, we characterize local GPF-rings.

Lemma 1.5. *A local ring R is GPF-ring if and only if every element a in R is either a non-zero-divisor or a nilpotent element.*

Proof. Assume that R is a local GPF-ring. Let $a \in R$. Since R is a GPF-ring, there exists a positive integer n such that $\operatorname{ann}_R(a^n)$ is pure. If $\operatorname{ann}_R(a^n) = 0$, then a is a non-zero-divisor. If $\operatorname{ann}_R(a^n) \neq 0$, there exists a non-zero $b \in \operatorname{ann}_R(a^n)$. So there exists $c \in \operatorname{ann}_R(a^n)$ such that $bc = b$. If $a^n \neq 0$, then $1 - c$ is a unit because R is local. Thus, $b = 0$, a contradiction. Consequently, $a^n = 0$, i.e. a is nilpotent.

Conversely, let $a \in R$. If a is a non-zero-divisor, then $\operatorname{ann}_R(a) = 0$ which is pure. If a is nilpotent, then there exists a positive integer n such that $a^n = 0$. So, $\operatorname{ann}_R(a^n) = R$ which is a pure ideal of R. Consequently, R is a GPF-ring.

Lemma 1.6. *Let R be a GPF-ring. If P is a prime ideal of R, then the localization,* R_P, *is a GPF-ring.*

Proof. Let $a/s \in R_P$. Since R is a GPF-ring, there exists a positive integer n such that a^nR is a flat R-module. But $(a/s)^n = a^nR_P$, so $(a/s)^nR_P$ is a flat R_P-module because flatness is a local property. Thus R_P is a GPF-ring.

So, we get the following corollary which was proved differently in Matlis [5].

Corollary 1.7. *A ring R is a PF-ring if and only if every localization*

R_P is an integral domain.

Proof. Assume R is a PF-ring. Taking $n=1$ in the proof of Lemma 1.6, we get the if direction.

Conversely, let $a \in R$, then aR_P is a flat R_P-module since R_P is an integral domain. Because flatness is a local property, aR is a flat R-module. So R is a PF-ring.

The following theorem characterizes GPF-rings through localizations.

Theorem 1.8. *A ring R is a GPF-ring if and only if for every $a \in R$ either a is a non-zero-divisor in each localization R_P or there exists a positive integer n such that $a^n=0$ in each R_P, where a is not a zero-divisor.*

Proof. Assume that R is a GPF-ring. Let $a \in R$, then there exists a positive integer n such that a^nR *is a* flat R-module. So a^nR_P is a flat R_P-module, i.e. $\mathrm{ann}_{R_P}(a^n)$ is a pure ideal in R_P. Exactly as in the proof of Lemma 1.5, either a is a non-zero-divisor in R_P or $a^n=0$ in R_P.

Conversely, assume that the condition holds. Let $a \in R$. If a is a non-zero-divisor in each R_P, then aR_P is a flat R_P-module for each P. Since flatness is a local property, aR is a flat R-module. If for some prime P, $a^n=0$ in R_P while for the others a is a non-zero-divisor, then a^nR_P is a flat R_P-module for all such prime ideals P of the first type. For all prime ideals of the second type, aR_P is a flat R_P-module. Consequently, a^nR is a flat R-module.

Theorem 1.9. *A ring R is a reduced GPF-ring if and only if R is a PF-ring.*

Proof. Clearly, every PF-ring is a GPF-ring. Also every PF-ring is reduced (without nontrivial nilpotent elements) see Al-Ezeh [1]. So R is a reduced PGF-ring.

Conversely, assume that R is a reduced GPF-ring. So for each prime ideal P, R_P is a reduced GPF-ring. That is R_P is an integral domain. By Corollary 1.7, R is a PF-ring.

More generally we prove the following theorem.

Theorem 1.10. *Let R be a GPF-ring. Then if N is the nilradical of R, R/N is a PF-ring.*

Proof. Let $a+N \in R/N$ and $b+N \in \mathrm{ann}_{R/N}(a+N)$. Then $ba \in N$. So, there exists a positive integer n such that $b^n a^n = 0$, i.e. $b^n \in \mathrm{ann}_R(a^n)$. Since R is a GPF-ring, there exists a positive integer m such that $\mathrm{ann}_R(a^m)$ is pure. By Lemma 1.3, $\mathrm{ann}_R(a^{nm})$ is pure. Since $b^n \in \mathrm{ann}_R(a^{nm})$, there exists $c \in \mathrm{ann}_R(a^{nm})$ such that $b^n c = b^n$. Hence $bc - b \in N$. Moreover, $ca \in N$, since $ca^{nm} = 0$. Thus $c+N \in \mathrm{ann}_{R/N}(a+N)$ and $(b+N)(c+N) = b+N$. Consequently, R/N is a PF-ring.

Theorem 1.11. *Let R be a GPF-ring. For any pure ideal I of R, R/I is a GPF-ring.*

Proof. Let $a+I \in R/I$. Since R is a GPF-ring, there exists a positive integer n such that $\mathrm{ann}_R(a^n)$ is pure. Now, we want to show that $\mathrm{ann}_{R/I}(a^n+I)$ is pure.. Let $x+I \in \mathrm{ann}_{R/I}(a^n+I)$, then $xa^n \in I$. Since I is pure, there exists $y \in I$ such that $xa^n y = xa^n$, i.e. $a^n(xy-x) = 0$. So, there exists $z \in \mathrm{ann}_R(a^n)$ such that $(xy-x)z = xy-x$. Thus, $xz - x \in I$. Hence $(x+I)(a^n+I) = I$ and $(x+I)(x+I) = x+I$. Therefore, $\mathrm{ann}_{R/I}(a^n+I)$ is pure. Consequently, R/I is a GPF-ring.

2. Generalized PF-rings and generalized PP-rings. Recall that a ring R is called a π-regular ring if for every $a \in R$, there exists a positive integer n such that $a^n = a^{2n}b$ for some $b \in R$, and a ring R is called quasi π-regular ring if the classical ring of quotient of R, $Q(R)$, is a π-regular ring. Hirano [4] proved that R is a quasi π-regular ring if and only if for each $a \in R$, there exists a positive integer n and a non-zero-divisor d such that $a^n d = a^{2n}$. The following theorem was proved by Hirano [4], but we give here an alternative proof using the characterization of GPF-rings via pure ideals.

Theorem 2.1. *A ring R is a GPP-ring if and only if it is a quasi π-regular, GPF-ring.*

Proof. Assume R is a GPP-ring, then it is a GPF-ring. Now, let $a \in R$. Then there exists a positive integer n and an idempotent e such that $\mathrm{ann}_R(a^n) = eR$. Then $a^n + e$ is a non-zero-divisor and $a^n(a^n+e) = a^{2n}$.

Conversely, assume R is a quasi π-regular, GPF-ring. Let $a \in R$. Since R is a quasi π-regular ring, there exists a positive integer n and a non-zero-divisor d such that $a^n d = a^{2n}$. Also, since R is a GPF-ring, there exists a positive integer m such that $\mathrm{ann}_R(a^m)$ is pure. Now, $a^{nm} d^m = a^{2nm}$.

Let $t = nm$ and $u = d^m$, then $a^t u = a^{2t}$ and u is a non-zero-divisor. By Lemma 1.3, $\mathrm{ann}_R(a^t)$ is pure. If $b = u = a^t$, then $b \in \mathrm{ann}_R(a^t)$. Since $\mathrm{ann}_R(a^t)$ is pure, there exists $e \in \mathrm{ann}_R(a^t)$ such that $be = b$. Now, consider $ue(1-e) = (u-a^t)e(1-e) = be(1-e) = 0$. Thus $e(1-e) = 0$. So e is an idempotent element. Clearly, $eR \subset \mathrm{ann}_R(a^t)$. Now, let $x \in \mathrm{ann}_R(a^t)$. Then $xa^t = 0$.
Consider

$$x(1-e)u = x(1-e)(u-a^t) = x(1-e)b = 0.$$

Thus $x(1-e) = 0$, i.e. $x = xe$. Therefore $\mathrm{ann}_R(a^t) = xe$. Hence R is a GPP-ring.

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