Math. J. Okayama Univ. 30 (1988), 199−211

SINGULAR POINT SETS OF A GENERAL CONNECTION AND BLACK HOLES

Dedicated to Professor Hisao Tominaga on his 60th birthday

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§ 1. **General connections and geodesics.** In the present paper the author will try to construct a theory of black holes as a subject in differential geometry by means of general connections, which are now called Otsuki connections mainly by Eastern European geometers, taking the results obtained in [10], [11] and the example in § 3 into consideration.

Let M^n be an n-dimensional manifold with a smooth general connection Γ, which we denote by (M^n, Γ). The concept of general connections was defined by the author in [5]. Let (P^i_j, Γ^i_{jk}) be the components of Γ in local coordinates u^i, i.e.

$$\Gamma = \partial u_i \otimes (P^i_j d^2 u^j + \Gamma^i_{jh} du^j \otimes du^h).$$

The part of the first order of Γ is represented as

$$P = \lambda(\Gamma) = \partial u_i \otimes P^i_j du^j,$$

which is a tensor field of type (1.1). A point of M^n is called a regular point of Γ if $\det(P^i_j) \neq 0$ and otherwise a singular one, and the set of all regular points is denoted by $\mathrm{reg}\,\Gamma$, which is open, and we set $\mathrm{sing}\,\Gamma = M^n - \mathrm{reg}\,\Gamma$.

A curve $\gamma: x = \gamma(t)$ for $a < t < b$ in M^n is called a geodesic of (M^n, Γ), if it satisfies the condition:

$$\frac{D}{dt}\frac{dx}{dt} = \psi(t) P\left(\frac{dx}{dt}\right),$$

where D denotes the covariant differentiation of Γ and $\psi(t)$ is a suitable function along γ, which is represented in local coordinates as

$$P^i_j \frac{d^2u^j}{dt^2} + \Gamma^i_{jh}\frac{du^j}{dt}\frac{du^h}{dt} = \psi P^i_j \frac{du^j}{dt}. \tag{1.1}$$

If (1.1) is satisfied with $\psi \equiv 0$, the parameter t is called an affine parameter of the geodesic, which is defined within an affine transformation for it. If we take a change of parameter $s = s(t)$, then (1.1) can be written as

$$\frac{D}{ds}\left(\frac{du^i}{ds}\right) = \left\{\left(\frac{dt}{ds}\right)^2 \psi + \frac{d^2 t}{ds^2}\right\} \frac{ds}{dt} P^i_j \frac{du^j}{ds}.$$

Therefore, integrating the differential equation

$$\left(\frac{dt}{ds}\right)^2 \psi + \frac{d^2 t}{ds^2} = 0,$$

we obtain an affine parameter s of the geodesic γ as

$$s = \int e^{\int \psi dt} dt.$$

On the other hand, taking a tensor field Q on M^n of type (1.1) with local components Q^i_j, consider the general connection $Q\Gamma$ with local components $(Q^i_k P^k_j, Q^i_k \Gamma^k_{jh})$. We see easily from (1.1) that γ is also a geodesic of $(M^n, Q\Gamma)$.

P can be considered as an endomorphism of the tangent space $T_x M^n$ at each point x of M^n, $P_x: T_x M^n \to T_x M^n$. On reg Γ, we denote the inverse of P by P^{-1}, then $P^{-1}\Gamma$ is a classical affine connection on reg Γ. Accordingly, an affine parameter s of a geodesic of (M^n, Γ) is also an affine parameter in the classical sense.

Definition. We call a curve $x = \gamma(s)$, $a \leqq s < b$, $-\infty < a < b \leqq +\infty$, of (M^n, Γ), a maximal semi-geodesic, ms-geodesic, if it is a geodesic of (M^n, Γ) for $a < s < b$, s is an affine parameter of this geodesic, and (a, b) is maximal on these properties with respect to b. We call a curve $x = \gamma(s)$, $a < s < b$, $-\infty \leqq a < b \leqq +\infty$, a maximal geodesic, m-geodesic, if it is a geodesic of (M^n, Γ) for $a < s < b$, with s as an affine parameter and (a, b) is maximal on these properties with respect to a and b.

Let (TM^n, M^n, π) be the tangent bundle over M^n. We consider now an open subset E of TM^n such that, for any point $x \in \pi(E)$, $E_x = E \cap T_x M^n$ is invariant under any scalar multiplication in $T_x M^n$. We say such E is a direction range of M^n and geodetically invariant, g-invariant, if it satisfies the following condition: For any maximal geodesic $x = \gamma(s)$, $a < s < b$, whose lift γ' in TM^n is not disjoint with E, then $\gamma' \subset E$.

In the following, we consider only such E and say E satisfies (α)-condition, if the following conditions hold:

i) For any point $p_0 \in \text{reg}\, \Gamma \cap \pi(E)$ and any ms-geodesic $x = \gamma(s)$ $(a \leqq s < b)$ with $p_0 = \gamma(a)$, $\gamma'(a) \in E$, it holds

$$\gamma(s) \to \operatorname{sing}\Gamma \text{ as } s \to b$$

or it diverges, i.e., for any compact set $K \subset M^n$, there exists s_0 such that $a < s_0 < b$ and $\gamma(s) \notin K$ for $s \geqq s_0$.

ii) For any m-geodesic $x = \gamma(s)$, $a < s < b$, such that $\gamma' \subset E$ and $\gamma(s) \to \operatorname{sing}\Gamma$ as $s \to a$ and also as $s \to b$, then $\gamma \subset \operatorname{sing}\Gamma$.

§ 2. **Black holes and sing Γ.** Let E be a g-invariant direction range of M^n. We call a geodesic γ of (M^n, Γ) an E-geodesic, if its lift γ' in TM^n lies in E.

Definition. $A \subset M^n$ is called a black hole of (M^n, Γ) with respect to E, if it has an open neighborhood U with the following properties:

i) ∂U is smooth and $\partial U \subset \operatorname{reg}\Gamma$,

ii) If an ms-E-geodesic $x = \gamma(s)$, $a \leqq s < b$, enters into U through ∂U at $\gamma(s_0)$, with $\gamma'(s_0) \notin T_{\gamma(s_0)}\partial U$, then $\gamma(s) \in U$ for $s > s_0$ and $\gamma(s)$ tends to A as $s \to b$.

iii) U does not contain divergent ms-E-geodesics.

U and ∂U in this definition are called a causal neighborhood and a causal boundary of A with respect to E respectively.

In the following, we assume the connection Γ satisfies the condition (α) for E. Let A be a black hole of (M^n, Γ) and U be a causal neighborhood of A with respect to E.

If an ms-E-geodesic $x = \gamma(s)$, $a \leqq s < b$, enters into U through ∂U at $\gamma(s_0)$, with $\gamma'(s_0) \notin T_{\gamma(s_0)}\partial U$, then the condition i) of (α) and the condition iii) of a black hole implies that $\gamma(s) \to \operatorname{sing}\Gamma \cap A$ as $s \to b$. Therefore, this fact tells us that under the condition (α) any black hole for E may be considered as a subset of sing Γ.

Take an ms-E-geodesic $x = \gamma(s)$, $0 \leqq s < b$, starting a point $p_0 = \gamma(0) \in (U-A) \cap \operatorname{reg}\Gamma$ and complete it to an m-geodesic $x = \gamma(s)$, $a < s < b$.

1) If $\gamma(s)$ $(0 \leqq s < b)$ is contained in U, then it tends to sing Γ by the condition iii) of a black hole and the condition i) of (α). Then, $x = \gamma_1(s) := \gamma(-s)$, $0 \leqq s < -a$, is an ms-E-geodesic. If γ_1 is contained in U, then $\gamma_1(s)$ also tends to sing Γ. The condition ii) of (α) implies that the m-geodesic $x = \gamma(s)$, $a < s < b$, is contained in sing Γ. This contradicts to $p_0 = \gamma(0) \in \operatorname{reg}\Gamma$. Hence, γ_1 must not be contained in U. γ_1 must run out U or tangent to ∂U at $\gamma_1(s_0)$, with $\gamma_1(s) \in U$ for $0 \leqq s < s_0$. If γ_1 runs

out U, with $\gamma_1'(s_0) \bar{\in} T_{\gamma_1(s_0)} \partial U$, then the ms-geodesic $x = \gamma(s)$, $-s_0 \leqq s < b$, enters into U through ∂U at $\gamma(-s_0)$, with $\gamma'(-s_0) \bar{\in} T_{\gamma(-s_0)} \partial U$, hence $\gamma(s)$ tends to A as $s \to b$, by the condition ii) of a black hole. If γ is tangent to ∂U, then must be on ∂U by the following Theorem 1, which contradicts to $p_0 \in U$.

2) If $\gamma(s)$ $(0 \leqq s < b)$ is not contained in U, then γ runs out U at a point $\gamma(s_0) \in \partial U$, with $\gamma'(s_0) \bar{\in} T_{\gamma(s_0)} \partial U$. Then, $\gamma(s)$ must tend to A as $s \to a$.

Theorem 1. *For (M^n, Γ), which satisfies the condition (α) for a g-invariant direction range E, a causal boundary of a black hole A with respect to E is totally geodesic with respect to E, i.e. any E-geodesic tangent to it at some point lies on it.*

Proof. Let U be a causal neighborhood of the black hole A. For any point $p_0 \in \partial U$ and an ms-E-geodesic $x = \gamma(s)$, $0 \leqq s \leqq b$, with $\gamma'(0) \in T_{p_0} \partial U$, we obtain $\gamma(s) \in \bar{U}$. In fact, we can take a family of ms-E-geodesics $\gamma_i(s)$, $0 \leqq s < b_i$, such that $\lim_{i\to\infty} \gamma_i'(0) = \gamma'(0)$ and $\gamma_i'(0)$ points to the inside of U at p_0, since $\partial U \subset \operatorname{reg} \Gamma$. We may suppose $\lim_{i\to\infty} b_i > b_0 > 0$. $\gamma_i(b_i) \in A$ and $\lim_{i\to\infty} \gamma_i(s) = \gamma(s)$ for $0 \leqq s \leqq b_0$, hence $s \leqq b$ and so $b_0 \leqq b$. Since $\gamma_i(s) \in U$ for $0 \leqq s \leqq b_0$, it must be $\gamma(s) \in \bar{U}$ for $0 \leqq s \leqq b_0$. By repeating this arguments for γ and using the condition ii) of a black hole, we see that $\gamma(s) \in \bar{U}$ for $0 \leqq s < b$.

On the other hand, taking a subsidiary Riemannian metric g on a neighborhood W of p_0 in $\operatorname{reg} \Gamma$, we can put that $\gamma'(0)$ and $\gamma_i'(0)$ are all unit vectors with respect to g and take b_0 uniformly for any ms-E-geodesic γ, with $\gamma'(0) \in T_p \partial U$, where $p = \gamma(0) \in W \cap \partial U$.

Now, take a geodesic $x = \gamma_0(s)$, $-c < s < c$, $c > 0$, such that $\gamma_0(0) = p_0$, s is an affine parameter, γ_0' is a unit vector with respect to g, $\gamma_0'(0) \bar{\in} T_{p_0} \partial U$, $\gamma_0(s) \in U$ for $-c \leqq s < 0$ and $\gamma_0(s) \bar{\in} \bar{U}$ for $0 < s \leqq c$. Taking another point $p_1 \in W \cap \partial U$ sufficiently near p_0, which can be joined with p_0 by an E-geodesic in W, we choose a geodesic $x = \gamma_1(s)$, $-c \leqq s \leqq c$, which satisfies the same conditions as γ_0. We consider the family of E-geodesics $x = \tau_s(t)$, $0 \leqq t \leqq 1$, such that $\tau_s(0) = \gamma_0(s)$ and $\tau_s(1) = \gamma_1(s)$ and t is an affine parameter for each geodesic τ_s. If c is sufficiently small, the construction of the family τ_s is always possible as Riemannian cases and we may assume that $\tau_s(t)$ is in W and differentiable with respect to s and t, and $\tau_c \subset M^n - \bar{U}$.

Let $s_0 \geqq 0$ be the value such that for $s > s_0$, $\tau_s \subset M^n - \bar{U}$ and $\tau_{s_0} \cap \partial U \neq \Phi$. If $s_0 > 0$, then take a point $q = \tau_{s_0}(t_0) \in \partial U$. We see easily that $0 < t_0 < 1$ and $\tau'_{s_0}(t_0) \in T_q\, \partial U$. By means of the above mentioned fact $\tau_{s_0} \subset \bar{U}$, which contradicts to $\tau_{s_0}(0) = \gamma_0(s_0) \bar{\in} \bar{U}$. Therefore it must be $s_0 = 0$.

If there exists t_0 $(0 < t_0 < 1)$ such that $\tau_0(t_0) \bar{\in} \partial U$, then $\tau_0(t_0) \bar{\in} U$, because $\tau_0(t_0) \in U$ implies $s_0 > 0$. Then, we can choose s_1 $(-c < s_1 < 0)$ such that $\tau_{s_1}(t_0) \bar{\in} U$ and τ_{s_1} passes through ∂U transversally at two points $\tau_{s_1}(t_1)$ and $\tau_{s_1}(t_2)$ with $0 < t_1 < t_0 < t_2 < 1$ and $\tau_{s_1}(t) \bar{\in} \bar{U}$ for $t_1 < t < t_2$. Let $x = \tau(t)$, $a < t < b$, be the m-E-geodesic such that $\tau(t) = \tau_{s_1}(t)$ for $0 \leqq t \leqq 1$. Then, by the condition i) of a black hole we obtain

$$\tau(t) \in U \text{ for } a < t < t_1 \text{ and } t_2 < t < b$$

and

$$\tau(t) \text{ tends to } A \cap \operatorname{sing} \Gamma \text{ as } t \to a \text{ or } t \to b.$$

By the condition ii) of (α), it must be $\tau \subset \operatorname{sing} \Gamma$, which contradicts $\tau(t_0) \in \operatorname{reg} \Gamma$. Hence we see that

$$\tau_0(t) \in \partial U \text{ for } 0 \leqq t \leqq 1. \qquad \text{Q. E. D.}$$

Theorem 2. *For (M^n, Γ), which satisfies the condition (α) for a g-invariant direction range E, let A be a black hole with respect to E and U a causal neighborhood of A, then one end of any m-E-geodesic through a regular point of U tends to $A \cap \operatorname{sing} \Gamma$ in U and the other end goes out of U through ∂U transversally.*

§ 3. An example.

Here, we shall consider the 4-manifold with a smooth general connection (R^4, Γ) studied in [11].

Let x, $i = 0, 1, 2, 3$, be the canonical coordinates of R^4, and t, r, θ, ϕ be the coordinates such that

$$x_0 = t,\ x_1 = r \sin\theta \cos\phi,\ x_2 = r \sin\theta \sin\phi,\ x_3 = r\cos\theta.$$

For the space-time metric g:

$$d\sigma^2 = -\left(1 - \frac{4m^2}{r^2}\right)dt^2 + \frac{2}{r}dt\,dr + r^2(d\theta^2 + \sin^2\theta\, d\phi^2) \tag{3.1}$$

given for $r \neq 0$, we can choose a smooth general connection Γ on R^4

which has the same system of geodesics as the connection determined by the Christoffel symbols made by g on $r \neq 0$, the symmetric affine connection which is metric with respect to g, denoted by Γ_g. (Theorem 2 in [11]).

The equations of a geodesic of Γ_g is

$$
(3.2) \qquad \begin{cases}
\dfrac{d^2 t}{ds^2} + \dfrac{4m^2}{r^2}\left(\dfrac{dt}{ds}\right)^2 - r^2\left(\dfrac{d\theta}{ds}\right)^2 - r^2 \sin^2\theta\left(\dfrac{d\phi}{ds}\right)^2 = 0, \\
\dfrac{d^2 r}{ds^2} + \dfrac{4m^2 B}{r}\left(\dfrac{dt}{ds}\right)^2 - \dfrac{8m^2}{r^2}\dfrac{dt}{ds}\dfrac{dr}{ds} - \dfrac{1}{r}\left(\dfrac{dr}{ds}\right)^2 - Br^3\left(\dfrac{d\theta}{ds}\right)^2 \\
\qquad - Br^3 \sin^2\theta\left(\dfrac{d\phi}{ds}\right)^2 = 0, \\
\dfrac{d^2\theta}{ds^2} + \dfrac{2}{r}\dfrac{dr}{ds}\dfrac{d\theta}{ds} - \cos\theta\sin\theta\left(\dfrac{d\phi}{ds}\right)^2 = 0, \\
\dfrac{d^2\phi}{ds^2} + \dfrac{2}{r}\dfrac{dr}{ds}\dfrac{d\phi}{ds} + 2\cot\theta\dfrac{d\theta}{ds}\dfrac{d\phi}{ds} = 0,
\end{cases}
$$

where $B = 1 - 4m^2/r^2$ and s is the canonical parameter of the geodesic as

$$
(3.3) \qquad \begin{aligned}
\frac{d\sigma^2}{ds^2} &= -\left(1 - \frac{4m^2}{r^2}\right)\left(\frac{dt}{ds}\right)^2 + \frac{2}{r}\frac{dt}{ds}\frac{dr}{ds} \\
&\quad + r^2\left[\left(\frac{d\theta}{ds}\right)^2 + \sin^2\theta\left(\frac{d\phi}{ds}\right)^2\right] \\
&= c = \begin{cases} -1 \\ 0 \\ 1 \end{cases}
\end{aligned}
$$

according to the sign of the right hand side of (3.1), which is an affine parameter.

Now, we denote the sets of $X = X^i\, \partial/\partial x^i$ with $r \neq 0$ such that $g_{ij}X^i X^j$ is negative, zero or positive by E_{-1}, E_0 and E_{+1}, respectively. The above fact shows that E_{-1} and E_{+1} are g-invariant dirtection ranges in the sense described in § 2 and E_0 is also g-invariant. TR^4 is disjoint sum of E_{-1}, E_{+1}, E_0 and $\pi^{-1}(r = 0)$.

For any geodesic γ, we may put $\theta \equiv \pi/2$ and have two constants A and J such that

$$
(3.4) \qquad \frac{1}{r}\left(\frac{dr}{ds} - Br\frac{dt}{ds}\right) = A,
$$

$$
(3.5) \qquad r^2\frac{d\phi}{ds} = J
$$

which, joining with (3.3), are equivalent to (3.2) (See § 1 of [9]). In the following, we shall discuss whether the set W $(r=0)$ is a black hole with the causal neighborhood $U(r<2m)$ or not for (R^4, Γ) with respect to $E=E_{-1}$ in the sense stated in § 2.

Let γ be a visible geodesic, i.e. $c=-1$ or 0, which enters into U, passing through ∂U transversally at $p_0=\gamma(0)$. Then we have

$$B=0 \text{ and } \frac{dr}{ds}=2mA<0 \text{ at } p_0.$$

From (3.4), (3.5) and (3.3) we obtain easily

$$\left(\frac{d\log r}{ds}\right)^2=A^2-B\left(\frac{J^2}{r^2}-c\right). \tag{3.6}$$

Case I: γ is visible, i.e. $c=-1$ or 0. We have

$$\frac{d\log r}{ds}<A \text{ and } r<2m \text{ for } s>0$$

and

$$r<2me^{As}, \tag{3.7}$$

from which we find

$$\lim_{s\to+\infty} r=0. \tag{3.8}$$

Then, from (3.4) we obtain

$$\frac{dt}{ds}=\frac{1}{B}\frac{d\log r}{ds}-\frac{A}{B}=\frac{1}{1-\dfrac{4m^2}{r^2}}\frac{d\log r}{ds}-\frac{A}{1-\dfrac{4m^2}{r^2}}$$

hence

$$t=t_0+\frac{1}{2}\log\frac{4m^2-r^2}{4m^2-r_0^2}+A\int_{s_0}^{s}\frac{r^2}{4m^2-r^2}ds \tag{3.9}$$

by integration, where $t_0=t(s_0)$, $r_0=r(s_0)$ and $s_0>0$. We obtain first from (3.9) the inequality:

$$\lim_{s\to+\infty} t<t_0+\frac{1}{2}\log\frac{4m^2}{4m^2-r_0^2}. \tag{3.10}$$

On the other hand, we have from (3.7)

$$e^{-2As}-1<\frac{4m^2}{r^2}-1,$$

and hence

$$\begin{aligned} t &> t_0+\frac{1}{2}\log\frac{4m^2-r^2}{4m^2-r_0^2}+A\int_{s_0}^{s}\frac{ds}{e^{-2As}-1} \\ &= t_0+\frac{1}{2}\log\frac{4m^2-r^2}{4\,m-r_0^2}-\frac{1}{2}\log\frac{\sinh|A|s}{\sinh|A|s_0}-\frac{A}{2}(s-s_0), \end{aligned}$$

i.e.

$$\begin{aligned} (3.11)\qquad t &> t_0+\frac{1}{2}\log\frac{4m^2-r^2}{4m^2-r_0^2}+\frac{1}{2}\log\sinh|A|s_0+\frac{1}{2}As_0 \\ &\quad -\frac{1}{2}\{\log\sinh|A|s-|A|s\}. \end{aligned}$$

Since we have for $x>0$

$$\log\sinh x-x=\log\frac{e^x-e^{-x}}{2}-x<\log\frac{e^x}{2}-x=-\log 2,$$

we obtain the inequality

$$(3.12)\qquad t>t_0+\frac{1}{2}\log\frac{4m^2-r^2}{4m^2-r_0^2}+\frac{1}{2}\log\sinh|A|s_0+\frac{1}{2}As_0+\log\sqrt{2}$$

for $s>s_0$, which implies

$$(3.13)\qquad \lim_{s\to+\infty}t\geqq t_0+\log\frac{2m}{\sqrt{4m^2-r_0^2}}+\frac{1}{2}\log\sinh|A|s_0+\frac{As_0}{2}+\log\sqrt{2}.$$

Case II: γ is non-visible, i.e. $c=1$. (3.6) becomes

$$(3.6')\qquad \left(\frac{d\log r}{ds}\right)^2=A^2+\left(\frac{4m^2}{r^2}-1\right)\left(\frac{J^2}{r^2}-1\right).$$

If $2m\leqq|J|$, we have for $0<r\leqq 2m$

$$(3.14)\qquad \left(\frac{d}{ds}\log r\right)^2\geqq A^2,$$

and so we can treat γ as the previous case and find that (3.7), (3.8), (3.10) and (3.13) also hold.

In the following, we suppose

$$(3.15) \qquad |J| < 2m.$$

For $r \leqq |J|$, (3.14) holds. If γ passes through the hypersurface $r = |J|$ at $p_1 = \gamma(s_1)$, $s_1 > 0$, then we have

$$(3.7') \qquad r < |J| e^{A(s-s_1)} \quad \text{for } s > s_1,$$

from which we find

$$(3.8') \qquad \lim_{s\to+\infty} r = 0.$$

We have also

$$(3.9') \qquad t = t_1 + \frac{1}{2}\log\frac{4m^2-r^2}{4m^2-J^2} + A\int_{s_1}^{s}\frac{r^2 ds}{4m^2-r^2},$$

where $t_1 = t(s_1)$ and which implies

$$(3.10') \qquad \lim_{s\to+\infty} t < t_1 + \frac{1}{2}\log\frac{4m^2}{4m^2-J^2}.$$

On the other hand, we have from (3.7')

$$\frac{r^2}{4m^2-r^2} < \frac{J^2}{4m^2 e^{2|A|(s-s_1)} - J^2}$$

and hence

$$\begin{aligned} t &> t_1 + \frac{1}{2}\log\frac{4m^2-r^2}{4m^2-J^2} + A\int_{s_1}^{s}\frac{J^2 ds}{4m^2 e^{2|A|(s-s_1)} - J^2} \\ &= t_1 + \frac{1}{2}\log\frac{4m^2-r^2}{4m^2-J^2} \\ &\quad -\frac{1}{2}\left[\log\left\{e^{|A|(s-s_1)} - \frac{J^2}{4m^2}e^{-|A|(s-s_1)}\right\}\right]_{s_1}^{s} - \frac{A}{2}(s-s_1) \\ &> t_1 + \frac{1}{2}\log\frac{4m^2-r^2}{4m^2-J^2} + \frac{1}{2}\log\left(1-\frac{J^2}{4m^2}\right) \\ &\quad -\frac{1}{2}[\log e^{|A|(s-s_1)} - |A|(s-s_1)], \end{aligned}$$

i.e.

$$(3.12') \qquad t > t_1 + \frac{1}{2}\log\frac{4m^2-r^2}{4m^2-J^2} + \frac{1}{2}\log\left(1-\frac{J^2}{4m^2}\right),$$

which implies

$$(3.13') \qquad \lim_{s\to+\infty} t \geqq t_1 + \log\frac{2m}{\sqrt{4m^2-J^2}} + \frac{1}{2}\log\left(1-\frac{J^2}{4m^2}\right).$$

Now, we investigate whether γ can attain the hypersurface $r = |J|$ in this case. We obtain from (3.6)

$$\left(\frac{dr}{ds}\right)^2 = r^2A^2 - B(J^2-r^2)$$
$$= \frac{1}{r^2}\{(A^2+1)r^4 - (4m^2+J^2)r^2 + 4m^2J^2\},$$

and

$$(3.16) \qquad s = \int_r^{2m} \frac{rdr}{\sqrt{(A^2+1)r^4-(4m^2+J^2)r^2+4m^2J^2}}$$
$$= \frac{1}{2}\int_{r^2}^{4m^2} \frac{dy}{\sqrt{(A^2+1)y^2-(4m^2+J^2)y+4m^2J^2}},$$

where $y = r^2$. Setting

$$f(y) := (A^2+1)y^2 - (4m^2+J^2)y + 4m^2J^2,$$

we find

$$f(J^2) = A^2J^4 < f(4m^2) = 16A^2m^4.$$

If we have

$$\frac{4m^2+J^2}{2(A^2+1)} \leqq J^2$$

i.e.

$$(3.17) \qquad 4m^2 \leqq J^2(2A^2+1),$$

then $f(y)$ is monotone increasing and positive in $J^2 \leqq y \leqq 4m^2$. Hence, s is monotone decreasing with respect to r in $|J| \leqq r \leqq 2m$, and so r is monotone decreasing from $2m$ to $|J|$. γ can attain to the hypersurface $r = |J|$.

Next, we consider the case

$$(3.18) \qquad 4m^2 > J^2(2A^2+1).$$

If the descriminant D of the quadratic function $f(y)$ is negative

$$D = (4m^2+J^2)^2 - 16m^2(A^2+1)J^2 = (4m^2-J^2)^2 - 16m^2A^2J^2 < 0$$

i.e.

$$4m^2 - J^2 < 4m|A||J|, \tag{3.19}$$

then we have $f(y) > 0$ for $J^2 \leqq y \leqq 4m^2$. We can also claim the same fact for γ as above. When $D = 0$, we have the same.

Finally we consider the case

$$4m^2 - J^2 > 4m|A||J| \tag{3.20}$$

under (3.18). Then, there exist two roots y_1, y_2 of $f(y) = 0$ such that

$$J^2 < y_1 < y_2 < 4m^2.$$

For $r_1 = \sqrt{y_1} < r < r_2 = \sqrt{y_2}$, (3.6) is impossible. Therefore, this argument is stopped. We have only the formula (3.16) for $r_2 \leqq r \leqq 2m$.

Therefore, we find that the exceptional geodesic γ is the one which satisfies the conditions:

$$\begin{cases} c = 1,\ (dr/ds)_{s=0} < 0,\ |J| < 2m, \\ 4m^2 > J^2(2A^2+1), \\ 4m^2 - J^2 > 4m|A||J|. \end{cases} \tag{3.21}$$

Setting

$$u = |dr/ds|_{s=0},\ v = |d\phi/ds|_{s=0},$$

we have from (3.4) and (3.5)

$$|J| = 4m^2 v,\ |A| = \frac{1}{2m}u.$$

(3.21) can be represented as

$$\begin{cases} u > 0,\ 0 \leqq v < 1/2m \\ 4m^2v^2 + 2u^2v^2 < 1, \\ 4m^2v^2 + 2uv < 1. \end{cases} \tag{3.21'}$$

From the last inequality of (3.21'), we find $2uv < 1$, and hence

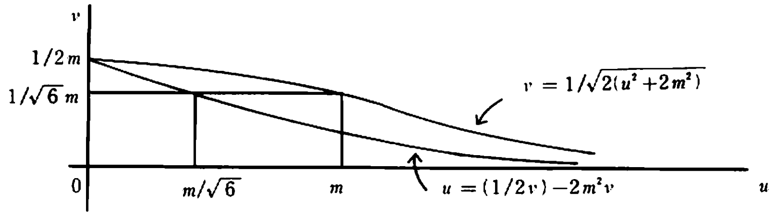
$$4m^2v^2 + 2u^2v^2 < 4m^2v^2 + 2uv < 1.$$

Therefore, (3.21') is equivalent to

$$0 < u < \frac{1}{2v} - 2m^2 v,\ 0 < v < \frac{1}{2m}$$

or

$$0 < u, \ v = 0.$$



Theorem 3. *Let (R^4, Γ) be the space with a smooth general connection Γ with the same system of geodesics determined by the metric g:*

$$d\sigma^2 = -\left(1 - \frac{4m^2}{r^2}\right)dt^2 + \frac{2}{r}dt\,dr + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

on $r \neq 0$. Any geodesic γ which enters into $U\,(r < 2m)$ through ∂U transversally at $\gamma(0)$ can not tend to $A\,(r = 0)$ if and only if

$$1 = -\left(1 - \frac{4m^2}{r^2}\right)\left(\frac{dt}{ds}\right)^2 + \frac{2}{r}\frac{dt}{ds}\frac{dr}{ds} + r^2\left\{\left(\frac{d\theta}{ds}\right)^2 + \sin^2\theta\left(\frac{d\phi}{ds}\right)^2\right\}$$

and

$$0 < u < \frac{1}{2v} - 2m^2 v, \ 0 < v < \frac{1}{2m} \text{ or } 0 < u, \ v = 0,$$

where

$$u = -\left(\frac{dr}{ds}\right)_{s=0}, \ v = \left\{\left(\frac{d\theta}{ds}\right)^2 + \sin^2\theta\left(\frac{d\phi}{ds}\right)^2\right\}^{1/2}_{s=0}.$$

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(Received July 10, 1987)