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HARMONIC REFLECTIONS ON SASAKIAN MANIFOLDS

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1. Introduction. Local geodesic symmetries on a Riemannian manifold are local diffeomorphisms which play an important role in the study of local Riemannian geometry. The properties of these transformations may be used to define some particular classes of Riemannian spaces. For example, it is well-known that a Riemannian manifold is locally symmetric if and only if all the local geodesic symmetries are isometries. Moreover, the following result is proved in [5]: *A Riemannian manifold is locally symmetric if and only if all local geodesic symmetries are harmonic maps.* We refer to [14] for other examples.

Local geodesic symmetries are local reflections with respect to a point. In [13], [16] this class of transformations has been generalized and the notion of a local reflection with respect to a submanifold, in particular with respect to a curve, has been introduced. In [16] T. J. Willmore and the second author obtained the following results (see also [13]):

Proposition 1. *Let σ: $[a,b] \to (M,g)$ be a topologically embedded curve in a Riemannian manifold (M,g). If the local reflection ϕ_σ with respect to σ is volume-preserving, then σ must be a geodesic. Moreover, a connected (M,g) is locally symmetric if and only if the reflections ϕ_σ with respect to all geodesics σ are volume-preserving. Finally, (M,g) is a space of constant curvature if and only if the reflections with respect to all geodesics are isometries.*

In [15] the following result is proved

Proposition 2. *Let σ: $[a,b] \to (M,g)$ be a topologically embedded curve in a Riemannian manifold (M,g). If the local reflection ϕ_σ with respect to σ is harmonic, then σ is a geodesic. Moreover, a connected Riemannian manifold is a space of constant curvature if and only if the local reflections with respect to all geodesics are harmonic.*

In the study of contact geometry, the Sasakian manifolds take a very special place. On these manifolds there are two particular nice and natural classes of geodesics, namely the integral curves of the characteristic vector field and the so-called ϕ-geodesics, that is, the geodesics which cut these integral curves orthogonally. Our main aim is to study here the local reflections with respect to these geodesics in the particular case that they are *harmonic maps*. In this way we find new characterizations of the so-called locally ϕ-symmetric spaces, the symmetric Sasakian manifolds and the three-dimensional Sasakian space forms.

2. Sasakian manifolds and locally ϕ-symmetric spaces. In this section we give some preliminaries about Sasakian manifolds. A smooth manifold M^{2n+1} is said to be an *almost contact manifold* if the structural group of its tangent bundle is reducible to $U(n)\times 1$. It is well-known that such a manifold admits a tensor field ϕ of type (1,1), a vector field ξ and a one-form η satisfying

$$\eta(\xi) = 1, \quad \phi^2 = -I + \eta \otimes \xi.$$

These conditions imply that $\phi\xi = 0$ and $\eta \circ \phi = 0$. Moreover, M admits a Riemannian metric g satisfying

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y)$$

for any tangent vector fields X and Y. This implies $\eta(X) = g(X, \xi)$. M together with these structure tensors (ϕ, ξ, η, g) is said to be an *almost contact metric manifold*.

Further, if these structure tensors satisfy

$$(1) \qquad (\nabla_X \phi) Y = g(X, Y)\xi - \eta(Y)X,$$

where ∇ denotes the Riemannian connection of g, M is said to be a *Sasakian manifold*. This condition (1) implies at once

$$(2) \qquad \nabla_X \xi = -\phi X,$$

from which it follows that ξ is a Killing vector field. Hence, the integral curves of ξ are geodesics. Moreover, a geodesic γ on a Sasakian manifold is said to be a *ϕ-geodesic* if $\eta(\gamma') = 0$. It is easy to see that a geodesic which is orthogonal to ξ at one point remains orthogonal to ξ.

Let R denote the Riemann curvature tensor defined by

$$R_{XY}Z = \nabla_{[X,Y]}Z - [\nabla_X, \nabla_Y]Z.$$

Then, on a Sasakian manifold, we have

$$(3) \qquad R_{XY}\xi = \eta(X)Y - \eta(Y)X.$$

For a Sasakian manifold local symmetry, i.e. $\nabla R=0$, is a very strong condition. Indeed we have

Proposition 3 ([10]). *A connected locally symmetric Sasakian manifold has constant sectional curvature* 1.

For this reason T. Takahashi [11] introduced the notion of a *locally ϕ-symmetric space* by requiring that on the Sasakian manifold we have

$$\phi^2(\nabla_V R)_{XY}Z=0$$

for all vector fields V, X, Y, Z orthogonal to ξ. This is equivalent to

$$(\nabla_V R)_{XYZW}=0$$

for all vector fields V, X, Y, Z, W orthogonal to ξ. We refer to [2],[3], [10] for several other useful characterizations of these spaces. Here we restrict to

Proposition 4 ([3]). *A Sasakian manifold is locally ϕ-symmetric if and only if*

$$\nabla_X R_{X\phi XX\phi X}=0$$

for all vector fields X orthogonal to ξ.

The simplest examples of locally ϕ-symmetric spaces are the so-called *Sasakian space forms*. They are defined as follows. A plane section in T_mM^{2n+1}, $m\in M$, is called a *ϕ-section* if it possesses an orthonormal basis of the form $\{X, \phi X\}$, where $X\in T_mM^{2n+1}$ is a vector orthogonal to ξ at m. The sectional curvature $K(X, \phi X)=H(X)=R(X, \phi X, X, \phi X)$ is called the associated *ϕ-sectional curvature*. A Sasakian manifold of constant ϕ-sectional curvature c is called a Sasakian space form. Its curvature tensor is given by

$$(4)\quad R_{XY}Z=\frac{c+3}{4}\{g(X,Z)Y-g(Y,Z)X\}$$
$$+\frac{c-1}{4}\{\eta(Y)\eta(Z)X-\eta(X)\eta(Z)Y-g(X,Z)\eta(Y)\xi+g(Y,Z)\eta(X)\xi$$
$$-g(Z,\phi Y)\phi X+g(Z,\phi X)\phi Y-2g(X,\phi Y)\phi Z\}.$$

Here we have the following useful criterion:

Proposition 5 ([12]). *A connected Sasakian manifold M of dimension $2n+1 \geqslant 5$ is a Sasakian space form if and only if, for every vector field X orthogonal to ξ, $R_{X\phi X}X$ is collinear with ϕX.*

We refer to [1],[17] for more details about contact geometry.

3. Reflections with respect to a curve. Let σ: $[a,b] \rightarrow (M,g)$ be a smooth embedded curve in a Riemannian manifold (M,g) and denote by U a tubular neighborhood of σ, i.e.

$$U = \{p \in M \mid \text{there exists a unique geodesic } \gamma \text{ of } M \text{ through } p \text{ cutting } \sigma \text{ orthogonally}\}.$$

Then, for any $p \in U$ we may put

$$p = \exp_{\sigma(t)}(ru),\ u \in T^{\perp}_{\sigma(t)}\sigma,\ \|u\| = 1,\ t \in [a,b],$$

where $r = d(p, \sigma(t))$.

The map ϕ_σ: $U \rightarrow U$ defined by

$$\phi_\sigma:\ p = \exp_{\sigma(t)}(ru) \mapsto \phi_\sigma(p) = \exp_{\sigma(t)}(-ru)$$

is a local diffeomorphism and is called a *local reflection with respect to* σ.

To describe the reflection ϕ_σ we use *Fermi coordinates*. We use the treatment developed in [9],[16]. Let σ be a unit speed curve ($\|\dot{\sigma}\| = 1$) and let $\{e_i,\ i = 1,\dots,n\}$ be an orthonormal basis of $T_{\sigma(a)}M$ such that $e_1 = \dot{\sigma}(a)$. In view of our main results we also suppose that σ is a *geodesic*. Next, let E_1 be the unit tangent field $\dot{\sigma}$ and $E_2,\dots,E_n$ the parallel normal vector fields along σ such that

$$E_i(a) = e_i,\ i = 2,\dots,n = \dim M.$$

Then, the Fermi coordinates $(x^1,\dots,x^n)$ with respect to $\sigma(a)$ and $(E_1,\dots,E_n)$ are defined by

$$x^1(\exp_{\sigma(t)} \sum_{j=2}^{n} t^j E_j) = t-a,$$

$$x^i(\exp_{\sigma(t)} \sum_{j=2}^{n} t^j E_j) = t^i,\ 2 \leqslant i \leqslant n.$$

For a vector $v \in T^{\perp}_{\sigma(t)}\sigma$ with $\exp_{\sigma(t)} v \in U$ we have

$$v = \sum_{a=2}^{n} x^a E_a(t) = ru$$

where $\|u\| = 1$ and $r^2 = \sum_{a=2}^{n} (x^a)^2$.

From this it follows that the local reflection ϕ_σ is given by

$$\phi_\sigma:\ (x^1, x^2, \dots, x^n) \mapsto (x^1, -x^2, \dots, -x^n).$$

In the next sections we shall express that the local reflection ϕ_σ with respect to the geodesic σ is harmonic. Therefore we need the expressions of the metric g and the inverse g^{-1} with respect to Fermi coordinates. Put

$$g_{ij} = g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right),\ i,j = 1, \dots, n.$$

Then we have, following the methods developed in [8], [9], [16]:

Proposition 6. *Let $m = \sigma(t)$ and $p = \exp_{\sigma(t)}(su)$, $\|u\| = 1$. We have*

$$(5)\qquad g_{11}(p) = 1 - s^2 R_{1u1u}(m) - \frac{1}{3}s^3 \nabla_u R_{1u1u}(m) + 0(s^4),$$

$$g_{1i}(p) = -\frac{2}{3}s^2 R_{1uiu}(m) - \frac{1}{4}s^3 \nabla_u R_{1uiu}(m) + 0(s^4),$$

$$g_{ij}(p) = \delta_{ij} - \frac{1}{3}s^2 R_{uiuj}(m) - \frac{1}{6}s^3 \nabla_u R_{uiuj}(m) + 0(s^4)$$

and

$$(6)\qquad g^{11}(p) = 1 + s^2 R_{1u1u}(m) + \frac{1}{3}s^3 \nabla_u R_{1u1u}(m) + 0(s^4),$$

$$g^{1i}(p) = \frac{2}{3}s^2 R_{1uiu}(m) + \frac{1}{4}s^3 \nabla_u R_{1uiu}(m) + 0(s^4),$$

$$g^{ij}(p) = \delta_{ij} + \frac{1}{3}s^2 R_{uiuj}(m) + \frac{1}{6}s^3 \nabla_u R_{uiuj}(m) + 0(s^4)$$

for $i,j = 2, \dots, n$.

Here we have posed $R_{uiuj}(m) = R_{uE_i(t)uE_j(t)}(\sigma(t))$, etc.

4. Harmonic reflections. Let (M, g) and (N, h) be two Riemannian manifolds with metrics g and h and let f: $(M, g) \to (N, h)$ be a smooth map. Then the covariant differential $\nabla(df)$ is called the *second fundamental form* and the *tension field* of f, denoted by $\tau(f)$, is the trace of $\nabla(df)$.

f is said to be *harmonic* if $\tau(f) = 0$.

To express this condition analytically, let $U \subset M$ be a domain with coordinates $(x^1,\dots,x^m)$ and $V \subset N$ a domain with coordinates $(y^1,\dots,y^n)$. Then f can be locally represented by $y^\alpha = f^\alpha(x^1,\dots,x^m)$, $\alpha = 1,\dots,n$. Further we have

$$\nabla(df)^\gamma_{ij} = \frac{\partial^2 f^\gamma}{\partial x^i \partial x^j} - {}^M\Gamma^k_{ij}\frac{\partial f^\gamma}{\partial x^k} + {}^N\Gamma^\gamma_{\alpha\beta}(f)\frac{\partial f^\alpha}{\partial x^i}\frac{\partial f^\beta}{\partial x^j}, \tag{7}$$

$i,j = 1,\dots,m$ and $\gamma = 1,\dots,n$. Here ${}^M\Gamma^k_{ij}$ and ${}^N\Gamma^\gamma_{\alpha\beta}$ respectively, denote the Christoffel symbols for (M,g) and (N,h) respectively. Hence, f is harmonic if and only if

$$\tau(f)^\gamma = g^{ij}(\nabla(df))^\gamma_{ij} = 0. \tag{8}$$

For more details about harmonic maps we refer to [6],[7].

From these remarks and from section 3 we now get easily

Proposition 7. *The local reflection ϕ_σ with respect to a curve σ is a harmonic map if and only if*

$$\begin{cases} \tau(\phi_\sigma)^1(p) = \{g^{11}\nabla(d\phi_\sigma)^1_{11} + 2g^{1i}\nabla(d\phi_\sigma)^1_{1i} + g^{ij}\nabla(d\phi_\sigma)^1_{ij}\}(p) = 0, \\ \tau(\phi_\sigma)^k(p) = \{g^{11}\nabla(d\phi_\sigma)^k_{11} + 2g^{1i}\nabla(d\phi_\sigma)^k_{1i} + g^{ij}\nabla(d\phi_\sigma)^k_{ij}\}(p) = 0 \end{cases} \tag{9}$$

for $i,j,k = 2,\dots,n$, where

$$\begin{aligned} \nabla(d\phi_\sigma)^1_{11}(p) &= -\Gamma^1_{11}(p) + \Gamma^1_{11}(\phi_\sigma(p)), \\ \nabla(d\phi_\sigma)^1_{1i}(p) &= -\Gamma^1_{1i}(p) - \Gamma^1_{1i}(\phi_\sigma(p)), \\ \nabla(d\phi_\sigma)^1_{ij}(p) &= -\Gamma^1_{ij}(p) + \Gamma^1_{ij}(\phi_\sigma(p)), \\ \nabla(d\phi_\sigma)^k_{11}(p) &= \Gamma^k_{11}(p) + \Gamma^k_{11}(\phi_\sigma(p)), \\ \nabla(d\phi_\sigma)^k_{1i}(p) &= \Gamma^k_{1i}(p) - \Gamma^k_{1i}(\phi_\sigma(p)), \\ \nabla(d\phi_\sigma)^k_{ij}(p) &= \Gamma^k_{ij}(p) + \Gamma^k_{ij}(\phi_\sigma(p)). \end{aligned} \tag{10}$$

It is worthwile to note that $\nabla(d\phi_\sigma)^1_{1i}$, $\nabla(d\phi_\sigma)^k_{11}$, $\nabla(d\phi_\sigma)^k_{ij}$ are even functions and $\nabla(d\phi_\sigma)^1_{11}$, $\nabla(d\phi_\sigma)^1_{ij}$, $\nabla(d\phi_\sigma)^k_{1i}$ are odd functions.

Using the well-known expression

$$\Gamma^k_{ij}(p) = \frac{1}{2}\sum_{l=1}^{n} g^{kl}\left\{\frac{\partial g_{il}}{\partial x^j} + \frac{\partial g_{jl}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^l}\right\}(p),$$

we can write down, using Proposition 6, power series expansions for the quantities given in (10). After a detailed computation which we omit here, we obtain

Proposition 8. *Let σ be a geodesic and $(x^1,\dots,x^n)$ the system of Fermi coordinates defined above. Then we have*

$$
\begin{aligned}
(11)\quad \Gamma^1_{11}(\phi_\sigma(p)) - \Gamma^1_{11}(p) &= 0(s^3),\\
\Gamma^i_{11}(\phi_\sigma(p)) + \Gamma^i_{11}(p) &= \frac{1}{3}s^2(\nabla_i R_{1u1u} + 2\nabla_u R_{1u1i} - 4\nabla_1 R_{1uiu})(m) + O(s^4),\\
\Gamma^1_{1i}(\phi_\sigma(p)) + \Gamma^1_{1i}(p) &= 0(s^2),\\
\Gamma^j_{1i}(\phi_\sigma(p)) - \Gamma^j_{1i}(p) &= -2sR_{1uij}(m) + 0(s^3),\\
\Gamma^1_{ij}(\phi_\sigma(p)) - \Gamma^1_{ij}(p) &= 2sR_{1jiu}(m) + 0(s^3),\\
\Gamma^k_{ij}(\phi_\sigma(p)) + \Gamma^k_{ij}(p) &= -\frac{1}{6}s^2(3\nabla_u R_{ikuj} + 3\nabla_u R_{uijk} + \nabla_k R_{uiuj})(m)\\
&\quad + 0(s^4),
\end{aligned}
$$

for all $i,j,k = 2,\dots,n$.

5. The main results. In this section we start with the consideration of local reflections with respect to the integral curves of the characteristic vector field ξ. These local reflections coincide with the local *ϕ-geodesic* symmetries introduced in [11] (see also [2]). Before giving our first result we state

Lemma 9 ([2],[11]). *A Sasakian manifold is locally ϕ-symmetric if and only if the local reflections with respect to the integral curves of the characteristic vector field ξ are isometries.*

Now we are ready to prove

Theorem 10. *A Sasakian manifold is locally ϕ-symmetric if and only if the local reflections with respect to the integral curves of the characteristic vector field ξ are harmonic.*

Proof. First, let M be a locally ϕ-symmetric space. Then the result follows from Lemma 9 since any isometry is harmonic.

Conversely, suppose that the local reflections with respect to the integral curves of ξ are harmonic. We use the techniques described in Section 3 and Section 4 with $m = \sigma(t) \in M^{2n+1}$ and $\dot\sigma(t) = \xi$. In this case it follows easily from the formulas (1), (2), (3) that

$$
\begin{aligned}
R_{1u1u}(m) &= 1,\ R_{1uiu}(m) = 0,\\
R_{1uij}(m) &= 0,\ \nabla_u R_{1u1u}(m) = 0
\end{aligned}
$$

for $i,j = 2,\dots,2n+1$ and u orthogonal to ξ at m. Hence, from (6),

$$g^{11}(p) = 1+s^2+0(s^4),$$
$$g^{1i}(p) = \frac{1}{4}s^3\nabla_u R_{1uiu}(m)+0(s^4)$$

and from (11)

$$\Gamma^j_{1i}(\phi_\sigma(p))-\Gamma^j_{1i}(p) = 0(s^3).$$

Further, we see from this and the second condition (9) that if ϕ_σ is harmonic, the vanishing of the coefficient of s^5 gives

$$(12) \qquad \sum_{i,j=2}^{2n+1} \nabla_u R_{uiuj}(3\nabla_u R_{ikuj}+3\nabla_u R_{uijk}+\nabla_k R_{uiuj}) = 0$$

for all $k = 2,\dots,2n+1$ and all points $m \in M$. Taking $E_k(t) = u$, (12) yields

$$\sum_{i,j=2}^{2n+1} (\nabla_u R_{uiuj})^2 = 0$$

for all u orthogonal to ξ. Hence

$$(13) \qquad \nabla_u R_{uiuj} = 0$$

and since ϕu is orthogonal to ξ, by putting $E_i(t) = E_j(t) = \phi u$ in (13) we get

$$\nabla_u R_{u\phi uu\phi u} = 0.$$

Now the desired result follows from Proposition 4.

Next, we consider local reflections with respect to ϕ-geodesics. We prove

Theorem 11. *Let M be a Sasakian manifold of dimension $\geqslant 5$. Then M has constant curvature 1 if and only if the local reflections with respect to all ϕ-geodesics are harmonic.*

Proof. If M is a space of constant curvature, then the local reflections with respect to any geodesic are harmonic (Proposition 2). In fact the reflections are isometries (Proposition 1).

Conversely, let σ be a ϕ-geodesic, $m = \sigma(t)$ and u a unit vector orthogonal to $\dot\sigma(t)$. Proceeding in the same way as before we now see after some calculation that the vanishing of the coefficient of s^3 in the second condition (9) leads to

$$\sum_{i=2}^{2n+1} R_{1uiu}R_{1uik} = 0 \tag{14}$$

for all u orthogonal to $\dot{\sigma}(t)$ and all $k = 2, \ldots, 2n+1$. Taking again $u = E_k(t)$, (14) yields

$$R_{1uiu} = 0. \tag{15}$$

Since the ϕ-geodesic σ is arbitrary we get from (15) that

$$R_{vuxu} = 0 \tag{16}$$

for all v orthogonal to the characteristic vector field ξ and all u, x orthogonal to v. So (16) implies at once that $R_{u\phi u}u$ must be collinear with ϕu for all u orthogonal to ξ. Hence, Proposition 5 implies that M is a Sasakian space form. Finally, for v orthogonal to u, x and ξ, (4) and (16) imply

$$0 = R_{vuxu} = -\frac{3}{4}(c-1)g(u, \phi v)g(x, \phi u). \tag{17}$$

Since dim $M \geqslant 5$, we can choose v orthogonal to ξ and x orthogonal to $\xi, v, \phi v$. By taking $u = \phi v + \phi x$, (17) yields $c = 1$. This completes the proof.

For three-dimensional manifolds we need

Lemma 12 ([4]). *Let M be a three-dimensional Sasakian space form. Then the local reflection with respect to any ϕ-geodesic is an isometry.*

We prove

Theorem 13. *A three-dimensional Sasakian manifold is a Sasakian space form if and only if the local reflections with respect to all ϕ-geodesics are harmonic.*

Proof. First, suppose M is a Sasakian space form. Then the result follows from Lemma 12.

Conversely, let σ be a ϕ-geodesic as in Theorem 11. Since dim $M = 3$, (14) is automatically satisfied. So we compute again the coefficient of

s^5 in the second condition (9). This gives

$$4\nabla_u R_{1u1u}(\nabla_i R_{1u1u}+2\nabla_u R_{1u1i}-4\nabla_1 R_{1uiu})$$
$$-\sum_{j,k=2}^{2n+1}\nabla_u R_{ujuk}(3\nabla_u R_{kiuj}+3\nabla_u R_{ukji}+\nabla_i R_{ukuj}) = 0.$$

Taking again $u = E_i(t)$ we get

$$12(\nabla_u R_{1u1u})^2+5\sum_{j,k=2}^{2n+1}(\nabla_u R_{ujuk})^2 = 0.$$

So

$$\nabla_u R_{1u1u} = 0 \tag{18}$$

for all u orthogonal to $E_1(t) = \dot{\sigma}(t)$. Since this must be true for all ϕ-geodesics, we get from (18)

$$\nabla_u R_{u\phi uu\phi u} = 0$$

for all u orthogonal to ξ. Hence M is locally ϕ-symmetric (Proposition 4). This implies that M has constant scalar curvature and so M is a Sasakian space form.

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