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AN ELEMENTARY PROOF OF A THEOREM OF MARGULIS FOR KLEINIAN GROUPS

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1. The purpose of the present paper is to give an elementary proof of a theorem of Margulis [6] for Kleinian groups, which is fundamental in the modern theory of Kleinian groups and 3-dimensional hyperbolic orbifolds ([7]).

An elementary proof of 2-dimensional version of the Margulis theorem was given by Marden [5].

Our proof consists of a series of lemmas concerning elliptic transformations and an application of Jørgensen's result [4]. If we consider only groups without torsion, then our proof would be very easy and quick. But, as in the case of Marden, we do not require groups to be finitely generated, geometrically finite, nor exclude torsions.

2. A Möbius transformation $z \mapsto (az+b)/(cz+d)$ $ad-bc=1$, is lifted from the extended complex plane $\hat{\mathrm{C}} = \mathrm{C} \cup \{\infty\}$ to a conformal homeomorphism of upper half 3-space $\mathrm{H}^3 = \{\mathbf{h} = z+t\mathbf{j} \mid z \in \mathrm{C},\ t>0\}$ (where $\mathbf{j}$ is one of quaternion units) onto itself. The extention is then $\mathbf{h} \mapsto \mathbf{h}' = (a\mathbf{h}+b)(c\mathbf{h}+d)^{-1}$.

A discrete subgroup G of the group of the Möbius transformations which preserve H^3 is called a *Kleinian group*. The set $\Lambda(G)$ of limit points lies in $\partial\mathrm{H}^3 = \hat{\mathrm{C}}$. The Kleinian groups for which *card* $\Lambda(G) \leqq 2$ are called *elementary*.

We will use the notation $D(p;\ r)$ for the (non-Euclidean) ball with radius r and center at p, and set

$$I(G;\ p;\ r) = \{g \in G:\ g(D(p;\ r)) \cap D(p;\ r) \neq \phi\}$$
$$G(p;\ r) = \text{subgroup of } G \text{ generated by } I(G;\ p;\ r).$$

What we are going to prove is the following:

Theorem (Margulis). *There exists a constant $r>0$ with the property that, for any point $p \in H^3$ and for any Kleinian group G, the subgroup $G(p;\ r)$ is elementary.*

If G leaves a disc $\Delta \in \partial\mathrm{H}^3$ invariant, then we call G a *Fuchsian* *group.*

Corollary (Marden [5]). *There exists a constant $r > 0$ with the property that, for any point $p \in H^3$ and for any Fuchsian group G, either $G(p; r)$ is cyclic, or*

$$G(p; r) = \langle h, g; h^2 = g^2 = id. \rangle$$

for some $h, g \in G(p; r)$.

3. As is well known, elementary Kleinian groups are classified into the following three types (see [2][3]).

Type 1 : $\Lambda(G) = \phi$.
In this case, G is a finite group.

Lemma 1 ([1]. Theorem 5.1). *For a Kleinian group G, the following conditions are equivalent:*
(i) *G is finite;*
(ii) *apart from id. G contains only elliptic element;*
(iii) *the elements of G have a common fixed point* **t** *in H^3.*

Type 2 : $\Lambda(G) = \{p\}$ $(p \in \hat{\mathbf{C}})$.
In this case, G contains a parabolic transfomation g with a fixed point $p \in \hat{\mathbf{C}}$.

Type 3 : $\Lambda(G) = \{p, q\}$, $p, q \in \hat{\mathbf{C}}$ *and* $p \neq q$.
In this case, G contains a loxodromic transformation g with the fixed points $p, q \in \hat{\mathbf{C}}$. Every element of G leaves $\{p, q\}$ invariant.
If $g \in G - \{id.\}$ is a non-parabolic element, the geodesic A_g in H^3 joining the fixed points of g is called the *axis* of g.

Lemma 2. *Let g and h be elliptic. If $G = \langle g, h \rangle$ is of Type 3, then*

$$g^2 = h^2 = id. .$$

Proof. By conjugation, we may assume that every element of G leaves $\{0, \infty\}$ invariant and is therefore of the form

$$z \mapsto \kappa z^s, \ \kappa \neq 0, s = \pm 1.$$

If

$$g(z) = \varkappa_1 z \text{ and } h(z) = \varkappa_2 z,\ |\varkappa_i| = 1\ (i = 1, 2),$$

then G is a finite cyclic group.

Suppose that

$$g(z) = \varkappa_1 z\ (|\varkappa_1| = 1), \text{ and } h(z) = \varkappa_2 z^{-1} = \varkappa_2/z,\ \varkappa_2 \neq 0.$$

Then the axis A_g of g is the j-axis and the axis A_h of h is the geodesic with end points $\sqrt{\varkappa_2}$ and $-\sqrt{\varkappa_2}$. Since A_g and A_h intersect at $\sqrt{|\varkappa_2|}\mathbf{j}$, g and h have a common fixed point $\sqrt{|\varkappa_2|}\mathbf{j}$ in H^3. By Lemma 1, G is of Type 1. Also, we obtain that

$$g(z) = \varkappa_1 z^{-1} = \varkappa_1/z \text{ and } h(z) = \varkappa_2 z^{-1} = \varkappa_2/z$$

for some $\varkappa_1, \varkappa_2 \in \mathbf{C} - \{0\}$. Thus $g^2 = h^2 = id.$.

The following condition for a finitely generated Kleinian group to be elementary is a direct consequence of the definition of the elementary groups in [2] p. 83, but will be useful in the sequel.

Lemma 3. *Let $G = \langle g_1, \ldots, g_n \rangle$ be a finitely generated Kleinian group.*

(a) *If there exists a point $p \in H^3 \cup \hat{\mathbf{C}}$ such that $g_i(p) = p$ for all $1 \leqq i \leqq n$, then G is elementary.*

(b) *If there exist the two distinct points $p, q \in \hat{\mathbf{C}}$ such that g_i leaves $\{p, q\}$ invariant for all $1 \leqq i \leqq n$, then G is elementary.*

4. In this section, we shall study groups generated by two elliptic transformations.

Lemma 4. *Let g and h be elliptic and suppose that a Kleinian group $G = \langle g, h \rangle$ is non-elementary. Then*

(a) *either gh or $ghg^{-1}h^{-1}$ is loxodromic;*

(b) *if $ghg^{-1}h^{-1}$ is loxodromic and if $g^2 \neq id.$, then*

$$\langle g, ghg^{-1}h^{-1} \rangle$$

is non-elementary.

Proof. Suppose that neither gh nor $ghg^{-1}h^{-1}$ are loxodromic. We may assume without loss of generality that

$$g(z) = \alpha z/\bar{\alpha} \qquad |\alpha| = 1,\ \alpha^2 \neq 1,$$

and

$$h(z) = (az+b)/(cz+d),\ ad-bc = 1.$$

The elements g, gh and $ghg^{-1}h^{-1}$ have real trace (up to signature)

$$\lambda = a+d \qquad \mu = \alpha a + \bar{\alpha} d$$
$$\gamma = 2 - bc(\alpha - \bar{\alpha})^2 = 2 + 2bc\,|\alpha - \bar{\alpha}|^2,$$

respectively. On solving this simultaneous equation with respect to a and d (as λ and μ are real) we find $a = \bar{d}$. Next, as $ghg^{-1}h^{-1}$ is not loxodromic, we have $\gamma \leqq 2$ and so $bc \leqq 0$. If $b = 0$, then $g(0) = h(0) = 0$. By Lemma 3(a), G is elementary, being a contradiction. In case $c = 0$, we have $g(\infty) = h(\infty) = \infty$, and a similar contradiction.

If h is such that $bc < 0$, we may select a positive t with $t^2 = |b/c|$ so that $b = -t^2\bar{c}$. Therefore

$$h(z) = (az - t^2\bar{c})/(cz + \bar{a}).$$

As

$$\begin{aligned} h(t\mathbf{j}) &= (at\mathbf{j} - t^2\bar{c})(ct\mathbf{j} + \bar{a})^{-1} \\ &= t(a\mathbf{j} - t\bar{c})(\overline{ct\mathbf{j} + \bar{a}})/(|ct|^2 + |a|^2) \\ &= t\mathbf{j}, \end{aligned}$$

we have $g(t\mathbf{j}) = h(t\mathbf{j}) = t\mathbf{j}$. By Lemma 1, G is of Type 1, and we have a contradiction.

Suppose that $H = \langle g, ghg^{-1}h^{-1}\rangle$ is elementary. As $ghg^{-1}h^{-1}$ is loxodromic, H is of Type 3, so $\langle g, hg^{-1}h^{-1}\rangle (= \langle g, ghg^{-1}h^{-1}\rangle)$ is of Type 3. Because g and $hg^{-1}h^{-1}$ are elliptic, Lemma 2 is applicable for $\langle g, hg^{-1}h^{-1}\rangle$. Therefore g is of order 2, and we arrive at a contradiction, too.

Lemma 5. *Let g and h be of order 2. If $g = \langle g, h\rangle$ is a Kleinian group, then G is elementary.*

Proof. Suppose that G is a non-elementary Kleinian group. By Lemma 4, gh or $ghg^{-1}h^{-1} = (gh)^2$ is loxodromic. Therefore gh is loxodromic.

Let $p, q \in \hat{\mathbf{C}}$ be the fixed points of gh. As $gh(p) = p$, we have $h(p) = g^{-1}(p) = g(p)$. Hence $gh(g(p)) = ghh(p) = g(p)$. Similarly we have $gh(g(q)) = g(q)$. Therefore an element g leaves $\{p, q\}$ invariant. By Lemma 3, $\langle g, gh\rangle = \langle g, h\rangle = G$ is elementary.

5. In this section, we investigate subgroups of finitely generated groups with some properties.

Lemma 6. Let $G = \langle g_1,\dots,g_n\rangle$ *be a Kleinian group such that all subgroups* $G_{i,j} = \langle g_i, g_j\rangle$ $(1 \leqq i, j \leqq n)$ *are elementary. If* G *is non-elementary, then* $g_1,\dots,g_n$ *are all elliptic.*

Proof. If g_1 is parabolic, then all $G_{1,j} = \langle g_1, g_j\rangle$ $(1 \leqq j \leqq n)$ are of Type 2. Let p be a parabolic fixed point of g_1. Since $g_j(p) = p$ for all j $(1 \leqq j \leqq n)$, by Lemma 3(a), G is elementary.

If g_1 is loxodromic, then all $G_{1,j} = \langle g_1, g_j\rangle$ $(1 \leqq j \leqq n)$ are of Type 3. Let p and q be the fixed points of g_1. Since all $g_j (1 \leqq j \leqq n)$ leave $\{p, q\}$ invariant, by Lemma 3(b), G is elementary.

Lemma 7. *Let* $G = \langle g_1,\dots,g_n\rangle$ *be a non-elementary Kleinian group. Suppose that all subgroups* $G_{i,j} (1 \leqq i, j \leqq n)$ *are elementary and that one of them is of Type 2. Then there exist* g_i, g_j *and* g_k $(1 \leqq i, j, k \leqq n)$ *such that the subgroup* $\langle g_k, g_i g_j g_i^{-1} g_j^{-1}\rangle$ *is non-elementary where* $g_i g_j g_i^{-1} g_j^{-1}$ *is parabolic.*

Proof. We may assume that $G_{1,2} = \langle g_1, g_2\rangle$ is of Type 2. By Lemma 6, g_1 and g_2 are elliptic. We may assume that g_1 and g_2 have a common fixed point ∞ and are therefore of the forms

$$g_1(z) = \alpha z/\bar{\alpha} \qquad |\alpha| = 1,\ \alpha^2 \neq 1,$$
$$g_2(z) = (az+b)/d \qquad ad = 1.$$

Then we have

$$g_1 g_2 g_1^{-1} g_2^{-1}(z) = z + ab(\alpha^2 - 1).$$

If $b = 0$, then $G_{1,2}$ is elliptic cyclic. Hence $ab(\alpha^2 - 1) \neq 0$, so $g_1 g_2 g_1^{-1} g_2^{-1}$ is parabolic.

Suppose that $\langle g_k, g_1 g_2 g_1^{-1} g_2^{-1}\rangle$ is elementary for every k $(1 \leqq k \leqq n)$. Then $\langle g_k, g_1 g_2 g_1^{-1} g_2^{-1}\rangle$ is of Type 2, and g_k fixes ∞. By Lemma 3(a), G is elementary.

Lemma 8. *Let* $G = \langle g_1,\dots,g_n\rangle$ *be a non-elementary Kleinian group. Suppose that all subgroups* $G_{i,j} = \langle g_i, g_j\rangle$ $(1 \leqq i, j \leqq n)$ *are elementary and that one of them is of Type 3. Then there exist* g_i, g_j *and* g_k $(1 \leqq i, j, k \leqq n)$ *such that the subgroup* $\langle g_k, g_i g_j g_i^{-1} g_j^{-1}\rangle$ *is non-elementary where* $g_i g_j g_i^{-1} g_j^{-1}$ *is loxodromic.*

Proof. We may assume that $G_{1,2}$ is of Type 3, and every element of

$G_{1,2}$ leaves $\{0, \infty\}$ invariant. By Lemma 6, g_1 and g_2 are elliptic. On applying Lemma 2, we have

$$g_i(z) = x_i/z \qquad x_i \neq 0 \ (\mathrm{i} = 1, 2)$$

(see the proof of Lemma 2). Therefore

$$g_1 g_2 g_1^{-1} g_2^{-1}(z) = x_1^2 z / x_2^2.$$

If $g_1 g_2 g_1^{-1} g_2^{-1}$ is not loxodromic, then $|x_1| = |x_2|$. The axes A_{g_1} and A_{g_2} intersect at $\sqrt{|x_1|}\mathbf{j}$, and so $g_1(\sqrt{|x_1|}\mathbf{j}) = g_2(\sqrt{|x_1|}\mathbf{j}) = \sqrt{|x_1|}\mathbf{j}$. By Lemma 1, $G_{1,2}$ is of Type 1; therefore, we arrive at a contradiction.

Suppose that $\langle g_k, g_1 g_2 g_1^{-1} g_2^{-1} \rangle$ is elementary for every k $(1 \leqq k \leqq n)$. Then $\langle g_k, g_1 g_2 g_1^{-1} g_2^{-1} \rangle$ is of Type 3, and g_k leaves $\{0, \infty\}$ invariant. By Lemma 3(b), G is elementary.

Lemma 9. *Let $G = \langle g_1, \ldots, g_n \rangle$ be a non-elementary Kleinian group and suppose that all $G_{i,j}$ $(1 \leqq i, j \leqq n)$ are of Type 1. Then there exist g_i, g_j and g_k such that either $\langle g_k, g_k g_i g_j \rangle$ or $\langle g_i, g_k g_i g_j \rangle$ is non-elementary where $g_k g_i g_j$ is loxodromic.*

Proof. First we shall show that for each g_k there exist $g_{i(k)}$ and $g_{j(k)}$ such that $g_k g_{i(k)} g_{j(k)}$ is loxodromic. Assume that $k = 1$. If every $G_{1,i}$ $(1 \leqq i \leqq n)$ is cyclic, then G is cyclic. Therefore, we may assume that $G_{1,2}$ is not cyclic. By Lemma 1, g_1 and g_2 have a unique common fixed point $\mathbf{t}_2$ in H^3. If every $G_{1,j}$ $(3 \leqq j \leqq n)$ is cyclic, then G is elementary. Hence there exists g_j such that g_1 and g_j have a unique common fixed point $\mathbf{t}_j$ $(\mathbf{t}_j \neq \mathbf{t}_2)$ in H^3. We put $j = 3$.

We may assume that

$$g_1(z) = \alpha z / \bar{\alpha} \quad \alpha^2 \neq 1, \ |\alpha| = 1.$$

Set

$$g_i(z) = (a_i z + b_i)/(c_i z + d_i) \quad a_i d_i - b_i c_i = 1 \ (i = 2, 3)$$

and

$$g_2 g_3(z) = (Az + B)/(Cz + D).$$

As $\{\mathbf{h} \in \mathrm{H}^3 \colon g_1(\mathbf{h}) = \mathbf{h}\} = \{t\mathbf{j} \colon t > 0\}$, we have $\mathbf{t}_i = t_i \mathbf{j}$ and $t_2 \neq t_3$. And the equation $g_i(t_i \mathbf{j}) = t_i \mathbf{j}$ implies that $b_i = -t_i^2 \bar{c}_i$. Hence a simple computation shows that

$$A = a_2a_3 - t_2^2\bar{c}_2c_3 \text{ and } D = -t_3^2c_2\bar{c}_3 + \bar{a}_2\bar{a}_3.$$

Now let us show that $g_1g_2g_3$ is loxodromic. If $g_1g_2g_3$ is not loxodromic, then g_2g_3 and $g_1(g_2g_3)$ have real traces. We have $A = \bar{D}$, so that

$$\bar{c}_2c_3(t_2^2 - t_3^2) = 0.$$

Since $t_2 \neq t_3$, this implies $c_2 = 0$ or $c_3 = 0$. In the case where $c_2 = 0$ $g_1(\infty) = g_2(\infty) = \infty$, therefore this contradicts the fact that g_1 and g_2 have a unique common fixed point $\mathbf{t}_2$. In the same way, if $c_3 = 0$, then we have a contradiction. Therefore $g_1g_2g_3$ is loxodromic.

Next, assume the lemma is false. For each g_k $(1 \leqq k \leqq n)$, there exist $g_{i(k)}$ and $g_{j(k)}$ such that $g_kg_{i(k)}g_{j(k)}$ is loxodromic. Then $\langle g_k, g_kg_{i(k)}g_{j(k)}\rangle$ and $\langle g_{i(k)}, g_kg_{i(k)}g_{j(k)}\rangle$ are of Type 3. As $\langle g_k, g_{i(k)}g_{j(k)}\rangle (= \langle g_k, g_kg_{i(k)}g_{j(k)}\rangle)$ is of Type 3 and as both g_k and $g_{i(k)}g_{j(k)}$ $(\in G_{i(k),j(k)})$ are elliptic, by Lemma 2, every g_k $(1 \leqq k \leqq n)$ is of order 2.

We may assume that

$$g_kg_{i(k)}g_{j(k)}(z) = \varkappa z \qquad |\varkappa| \neq 1,\ \varkappa \neq 0,$$

and (by the proof of Lemma 2)

$$g_k(z) = \varkappa_k/z \qquad \varkappa_k \neq 0.$$

As $g_{i(k)}$ is of order 2, and as $\langle g_{i(k)}, g_kg_{i(k)}g_{j(k)}\rangle$ is of Type 3, we have

$$g_{i(k)}(z) = -z \text{ or } g_{i(k)}(z) = \varkappa_{i(k)}/z\ (\varkappa_{i(k)} \neq 0).$$

If $g_{i(k)}(z) = -z$, then

$$\begin{aligned} g_kg_{j(k)}(z) &= g_k(g_{i(k)}(g_k(g_kg_{i(k)}g_{j(k)})))(z) \\ &= -\varkappa z. \end{aligned}$$

As $G_{k,j(k)}$ is of Type 1, $g_kg_{j(k)}$ is elliptic. Hence we have $|\varkappa| = 1$, this contradict our assumption.

Let $g_{i(k)}(z) = \varkappa_{i(k)}/z$. As $g_kg_{i(k)}(z) = \varkappa_kz/\varkappa_{i(k)}$ is elliptic, we have $|\varkappa_k| = |\varkappa_{i(k)}|$. Because

$$g_{j(k)}(z) = g_{i(k)}(g_k(g_kg_{i(k)}g_{j(k)}))(z) = \varkappa_{i(k)}\varkappa z/\varkappa_k$$

and is elliptic, we have $|\varkappa_{i(k)}\varkappa/\varkappa_k| = |\varkappa| = 1$. This contradicts our assumption $|\varkappa| \neq 1$.

Remark. By Lemmas 6–9, if $G = \langle g_1, \ldots, g_n\rangle$ $(n \geqq 3)$ is a non-elementary Kleinian group, then there exsist i, j, k $(1 \leqq i, j, k \leqq n)$ such that $\langle g_i, g_j, g_k\rangle$ is non-elementary.

6. Before proving the theorem we need three more lemmas.

Lemma 10. *Let* $\{g_n\}_{n\in\mathbf{N}}$ *be a sequence of Möbius transformations with the property that there exist a point* $p \in H^3$ *and a number* $r > 0$ *such that*

$$g_n(D(\mathbf{j}; r)) \cap D(\mathbf{j}; r) \neq \phi$$

Then there is a subsequence converging to a Möbius transformation.

Proof. We may assume that $p = \mathbf{j}$. If $g_n(z) = (a_n z + b_n)/(c_n z + d_n)$ $(a_n d_n - b_n c_n = 1)$ are such that $g_n(D(\mathbf{j}; r)) \cap D(\mathbf{j}; r) \neq \phi$, then $d(\mathbf{j}, g_n(\mathbf{j})) < 2r$, where $d(.,.)$ is the hyperbolic distance in H^3. By [1], [2], we have

$$\|g_n\|^2 = |a_n|^2 + |b_n|^2 + |c_n|^2 + |d_n|^2 = 2\cosh\,[d(\mathbf{j}, g_n(\mathbf{j}))] < 2\cosh\,(2r),$$

a conclusion.

Lemma 11 ([4], Lemma 2). *Let* $\{g_n\}_{n\in N}$ *and* $\{h_n\}_{n\in N}$ *be two sequences of Möbius transformations converging to Möbius transformations* g *and* h, *respectively. Suppose that, for each* $n \in \mathbf{N}$ *the group* $\langle g_n, h_n\rangle$ *is discrete and non-elementary. Then the following is true*;

(a) *g is not the identity.*

(b) *If g is elliptic, then the orders of g_n are constant for all large indices.*

The proof of following lemma is similar to the Marden's proof of our result for Fuchsian groups [5].

Lemma 12. *There exists an* $r > 0$ *with the following property: Given any point* $p \in H^3$ *and any Kleinian group* G, *if*

$$I(G; p; r) = \{g_1, \ldots, g_m\}$$

holds, then all $G_{i,j} = \langle g_i, g_j\rangle$ $(1 \leqq i, j \leqq m)$ *are elementary.*

Proof. We may assume that $p = \mathbf{j}$. Assume that the conclusion of the lemma is false. There is a group G_n which contains elements g_n, h_n

$\in I(G_n;\ \mathrm{j};\ 1/n)$ such that the subgroup $H_n = \langle g_n, h_n\rangle$ is non-elementary.

By Lemma 10, for subsequences, which are denoted by $\{g_n\}$ and $\{h_n\}$ as well, $g = \lim_{n\to\infty} g_n$ and $h = \lim_{n\to\infty} h_n$ exist. As a consequence of Lemma 11(a), neither g nor h is the identity. We have $g(\mathrm{j}) = \mathrm{j}$ and $h(\mathrm{j}) = \mathrm{j}$, so that g and h are elliptic. Hence by Lemma 11(b), for all large indices, g_n and h_n have the constant order μ, ν respectively. By Lemma 5, if $\mu = \nu = 2$, then H_n is elementary.

If $\mu \neq 2$, by Lemma 4, H_n contains a subgroup H'_n or H''_n with the following properties:

(a) $H'_n = \langle g_n, g_n h_n\rangle$ is non-elementary and $g_n h_n$ is loxodromic;

(b) $H''_n = \langle g_n, g_n h_n g_n^{-1} h_n^{-1}\rangle$ is non-elementary and $g_n h_n g_n^{-1} h_n^{-1}$ is loxodromic.

If H'_n appears for an infinitely many n, then for a subsequence (with the same indices, for simplicity) $gh = \lim_{n\to\infty} g_n h_n$ exists. As a consequence of Lemma 11(a), gh is not the identity. We have $gh(\mathrm{j}) = \mathrm{j}$. Hence by Lemma 11(b), for all large indices $g_n h_n$ are elliptic. This fact contradicts property (a).

If H''_n appears for an infinite number of n, then a subsequence with $ghg^{-1}h^{-1} = \lim_{n\to\infty} g_n h_n g_n^{-1} h_n^{-1}$ exists. As a consequence of Lemma 11(a), $ghg^{-1}h^{-1}$ is elliptic. Hence by Lemma 11(b), for all large indices $g_n h_n g_n^{-1} h_n^{-1}$ are elliptic. This fact contradicts property (b).

For $\nu \neq 2$, the argument is completely the same.

Proof of Theorem. Assume that the conclusion of Theorem is false. We may assume that $p = \mathrm{j}$ and that for each n, there is a group G_n such that $G_n(\mathrm{j};\ 1/n)$ is non-elementary. We consider only such $1/n$ that satisfies Lemma 12.

Set $H_n = G_n(\mathrm{j};\ 1/n)$ and $I(G_n;\ \mathrm{j};\ 1/n) = \{g_{n,1},\dots,g_{n,m}\}$. In accordance with Lemma 12, each $(H_n)_{i,j} = \langle g_{n,i}, g_{n,j}\rangle$ is elementary. By Lemma 6, for all $1 \leqq i, j \leqq m$ $g_{n,i}$ are elliptic. As a consequence of Lemma 7, Lemma 8, and Lemma 9, the group H_n contains one of the following non-elementary subgroup H'_n, H''_n, H'''_n:

(a) $H'_n = \langle g_{n,k}, g_{n,i} g_{n,j} g_{n,i}^{-1} g_{n,j}^{-1}\rangle$ and $g_{n,i} g_{n,j} g_{n,i}^{-1} g_{n,j}^{-1}$ is non-elliptic;

(b) $H''_n = \langle g_{n,k}, g_{n,k} g_{n,i} g_{n,j}\rangle$ and $g_{n,k} g_{n,i} g_{n,j}$ is loxodromic;

(c) $H'''_n = \langle g_{n,i}, g_{n,k} g_{n,i} g_{n,j}\rangle$ and $g_{n,k} g_{n,i} g_{n,j}$ is loxodromic.

Clearly one of the cases (a),(b), and (c) occurs for an infinitely many n. In any case, we can obtain a contradiction by the same argument as in the latter half of the proof of Lemma 12.

7. *Proof of Corollary.* If r is a positive number given in Theorem, then $G(p;\, r)$ is elementary. An elementary Fuchsian group is known to be cyclic or $\langle h, g;\, h^2 = g^2 = id.\rangle$ for some h, g.

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