

OPERATIONS ASSOCIATED WITH THE G -EQUIVARIANT UNITARY COBORDISM THEORY

MICHIKAZU FUJII

Introduction. Let G be a compact abelian Lie group. In the previous paper [4] we have introduced a G -equivariant cohomology theory which is concerned with the G -equivariant unitary cobordism theory. In the equivariant cohomology theory there is the splitting principle and Chern classes are defined for complex G -vector bundles.

In this paper we shall study on cohomology operations in the equivariant cohomology theory. In §1 we consider Landweber-Novikov operations in our equivariant cohomology theory. And, in §2 we observe mod p Steenrod operations in the G -equivariant unitary cobordism theory and research on them in connection with the Landweber-Novikov operations introduced in §1.

1. Landweber-Novikov operations. Let G be a compact abelian Lie group. Let $U_c^*(-)$ and $K_c^*(-)$ be the G -equivariant unitary cobordism theory and the G -equivariant complex K -theory, respectively. By making use of Thom classes in K_c^* -theory, we can get a natural multiplicative transformation

$$\mu_c: U_c^*(-) \rightarrow K_c^*(-)$$

of the cohomologies (cf. [3], [4]). We take up a multiplicative set T_K consisting of all one dimensional representations in the representation ring $R(G) \cong K_c(\text{pt})$ and we consider a multiplicative system $T = \mu_c^{-1}(T_K)$ in U_c^0 . Then our multiplicative G -equivariant cohomology theory $h_c^*(-)$ is defined by

$$h_c^*(-) = T^{-1}U_c^*(-),$$

where $T^{-1}U_c^*(-)$ means a ring localized by the multiplicative system T .

Using the local triviality of complex G -vector bundles [5] and Theorem 4.5 in [4] we obtain the following splitting principle:

Proposition 1.1. *Let ξ be an n -dimensional complex G -vector bundle over a compact G -space X . Then there exist a compact G -space $F(\xi)$, a G -map $\pi: F(\xi) \rightarrow X$ and n complex G -line bundles ξ_1, \dots, ξ_n over $F(\xi)$ satisfying the following conditions:*

- 1) $\pi^* : h_c^*(X) \rightarrow h_c^*(F(\xi))$ is a monomorphism.
- 2) $\pi^*\xi$ is isomorphic to the sum $\xi_1 \oplus \cdots \oplus \xi_n$.

Proposition 1.2. *Let ξ and η be n and m -dimensional complex G -vector bundles over a compact G -space X , respectively. Then there exist a compact G -space F and a G -map $\pi : F \rightarrow X$ satisfying the following conditions :*

- 1) $\pi^* : h_c^*(X) \rightarrow h_c^*(F)$ is a monomorphism.
- 2) $\pi^*(\xi)$ and $\pi^*(\eta)$ are isomorphic to the sums of n and m G -line bundles over F , respectively.

Furthermore we have G -equivariant Chern classes $c_i^G(\xi) \in h_c^{2i}(X)$, $0 \leq i \leq n$ ($c_0^G(\xi) = 1$), of an n -dimensional complex G -vector bundle ξ over a compact G -space X [4].

We now define Landweber-Novikov operations [1, 9] in the cohomology theory $h_c^*(-)$ and call up their basic properties. Let $\mathbf{t} = (t_1, t_2, \dots)$ be a sequence of indeterminates. Assigning $\deg t_i = -2i$ for each $i \geq 1$, $U_c^*(-)[[\mathbf{t}]]$ and $h_c^*(-)[[\mathbf{t}]]$ become multiplicative G -equivariant cohomology theories.

Let ξ be an n -dimensional complex G -vector bundle over a compact G -space X and let $\pi : F(\xi) \rightarrow X$ and ξ_1, \dots, ξ_n be ones of Proposition 1.1. Consider the following

$$\prod_{i=1}^n (1 + e(\xi_i)t_1 + \cdots + e(\xi_i)^k t_k + \cdots) \in U_c^*(F(\xi))[[\mathbf{t}]]$$

and

$$(1) \quad \prod_{i=1}^n (1 + c_1^G(\xi_i)t_1 + \cdots + c_1^G(\xi_i)^k t_k + \cdots) \in h_c^*(F(\xi))[[\mathbf{t}]],$$

where $e(\xi_i) \in U_c^2(F(\xi))$ is the Euler class of ξ_i and $c_1^G(\xi_i) = \frac{e(\xi_i)}{1} \in h_c^2(F(\xi))$ is the first Chern class of ξ_i .

Given an n -tuple $\iota = (i_1, \dots, i_n)$ of non-negative integers, denote by

$$\sum x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n}$$

the least symmetric polynomial in variables x_1, \dots, x_n which contains the term $x_1^{i_1} \cdots x_n^{i_n}$. The symmetric polynomial can be written as a polynomial $P(\sigma_1, \dots, \sigma_n)$ in the elementary symmetric functions $\sigma_1, \dots, \sigma_n$ of the variables x_1, \dots, x_n :

$$P_i(\sigma_1, \dots, \sigma_n) = \sum x_1^{i_1} x_2^{i_2} \dots x_n^{i_n}.$$

As for the coefficient of $t_{i_1} t_{i_2} \dots t_{i_n}$ in the sequence (1), by making use of the splitting principle, we get the equality

$$\begin{aligned} \sum c_1^G(\xi_1)^{i_1} c_1^G(\xi_2)^{i_2} \dots c_1^G(\xi_n)^{i_n} &= P_i(\sigma_1, \dots, \sigma_n) \\ &= \pi^* P_i(c_1^G(\xi), \dots, c_n^G(\xi)), \end{aligned}$$

where $\sigma_k = \sigma_k(c_1^G(\xi_1), \dots, c_n^G(\xi_n))$ is the k -th elementary symmetric function of the classes $c_1^G(\xi_1), \dots, c_n^G(\xi_n)$.

Let us define the *total Chern class* $c_i(\xi)$ of ξ in the theory $h_G^*(-)[[\mathbf{t}]]$ by

$$c_i(\xi) = \sum_j P_j(c_1^G(\xi), \dots, c_n^G(\xi)) \mathbf{t}_i \in h_G^*(X)[[\mathbf{t}]]$$

where $\mathbf{t}_i = t_{i_1} t_{i_2} \dots t_{i_n}$. Then, in virtue of the naturality of Euler classes of G -line bundles, the splitting principle and the external product we obtain

Proposition 1.3. *The total Chern classes satisfy the following properties:*

- (1) (*naturality*) $c_i(f^*(\xi)) = f^*(c_i(\xi))$.
- (2) (*multiplicativity*) $c_i(\xi \times \eta) = c_i(\xi) \times c_i(\eta)$.
- (3) (*normality*) $c_i(\varepsilon) = 1$,

where $\varepsilon : \text{pt} \times C \rightarrow \text{pt}$ is the trivial G -line bundle over a point.

- (4) *If ξ is a complex G -line bundle, then*

$$c_i(\xi) = 1 + c_1^G(\xi)t_1 + \dots + c_i^G(\xi)t_k + \dots$$

Let $T(\xi)$ be the Thom space of an n -dimensional complex G -vector bundle ξ over a compact G -space X . Then, by making use of the Thom isomorphism

$$\phi(\xi) : h_G^*(X) \rightarrow \tilde{h}_G^{*+2n}(T(\xi)),$$

we obtain the Thom isomorphism

$$\phi_i(\xi) : h_G^*(X)[[\mathbf{t}]] \rightarrow \tilde{h}_G^{*+2n}(T(\xi))[[\mathbf{t}]],$$

which is defined by

$$\phi_i(\xi)(\sum a_{i_1, \dots, i_k} t_1^{i_1} \dots t_k^{i_k}) = \sum \phi(\xi)(a_{i_1, \dots, i_k}) t_1^{i_1} \dots t_k^{i_k}.$$

Put

$$s_i(\xi) = \phi_i(\xi)(c_i(\xi)) \in \tilde{h}_G^*(T(\xi))[[\mathbf{t}]].$$

Then we have

Proposition 1.4. *The classes $s_i(\)$ satisfy the following properties :*

- (1) (*naturality*) $s_i(f^*(\xi)) = f^*(s_i(\xi))$.
- (2) (*multiplicativity*) $s_i(\xi \times \eta) = s_i(\xi) \times s_i(\eta)$.
- (3) (*normality*) $s_i(\varepsilon) = t_h(\varepsilon) \in \tilde{h}_c^2(S^2)$.
- (4) *If ξ is a complex G -line bundle, then*

$$s_i(\xi) = t_h(\xi) + t_h(\xi)^2 t_1 + \cdots + t_h(\xi)^k t_{k-1} + \cdots$$

where $t_h(\xi)$ is the Thom class of ξ in the theory $h_c^*(-)$.

Let γ_c^n be the universal complex G -vector bundle and denote by $M_n(G)$ the Thom space of γ_c^n . Let W be a complex G -module and $G_n(W)$ the Grassmann manifold of complex n -planes. Then γ_c^n and $M_n(G)$ are the limit of the canonical n -dimensional G -vector bundle

$$\gamma_c^n(W) = (E_n(W), \pi, G_n(W))$$

and the Thom space $M_n(W) = T(\gamma_c^n(W))$, respectively.

Let $x \in U_c^{2n}(X)$ be represented by $f: V^c \wedge X^+ \rightarrow M_{1V^c+n}(W) \subset M_{1V^c+n}(G)$, where $X^+ = X \cup \{\infty\}$ (disjoint union), V^c means the one point compactification of a complex G -module V and $\|V\| = \dim_c V$. Defining

$$s_i: U_c^*(X) \rightarrow h_c^*(X)[[\mathbf{t}]]$$

by

$$s_i(x) = \phi_i(V)^{-1} f^*(s_i(\gamma_c^{1V^c+n}(W))),$$

we obtain a natural transformation

$$s_i: U_c^*(-) \rightarrow h_c^*(-)[[\mathbf{t}]]$$

of G -equivariant cohomology theories.

Proposition 1.5. *The natural transformation s_i has the following properties :*

- (1) (*naturality*) $s_i(g^*(x)) = g^*(s_i(x))$.
- (2) (*multiplicativity*) $s_i(xy) = s_i(x) s_i(y)$.
- (3) (*normality*) i) $s_i(t(\xi)) = s_i(\xi)$ for the Thom class $t(\xi) \in \tilde{U}_c^{2n}(T(\xi))$ of an n -dimensional complex G -vector bundle ξ , ii) $s_i(1) = 1$, and iii) $s_i(V) = t_h(V)$.

Let $\omega = (\omega_1, \omega_2, \dots)$ be a sequence of non-negative integers with $\omega_i = 0$ except for a finite number of terms, $|\omega| = \sum \omega_i$ and $\underline{t}^\omega = t_1^{\omega_1} t_2^{\omega_2} \dots$. Put

$$s_t(x) = \sum_{\omega} s_{\omega}(x) \underline{t}^{\omega}$$

for $x \in U_c^*(X)$. Then, from the properties of s_t it follows

Theorem 1.6. *For each sequence $\omega = (\omega_1, \omega_2, \dots)$ there exists an operation*

$$s_{\omega} : U_c^*(-) \rightarrow h_c^{*+2|\omega|}(-)$$

with the following properties :

- (1) (natural) $s_{\omega}(g^*(x)) = g^*(s_{\omega}(x))$.
- (2) (multiplicative) $s_{\omega}(xy) = \sum_{\alpha+\beta=\omega} s_{\alpha}(x) s_{\beta}(y)$

where $\alpha + \beta = (\alpha_1 + \beta_1, \alpha_2 + \beta_2, \dots)$.

(3) $s_0(x) = \frac{x}{1}$ for $0 = (0, 0, \dots)$.

(4) (stable) $s_{\omega}\sigma(V) = \sigma_h(V) s_{\omega}$, where $\sigma(V)$ and $\sigma_h(V)$ are suspension isomorphisms in the theories $U_c^*(-)$ and $h_c^*(-)$.

(5) If ξ is 1-dimensional, then

$$s_{\alpha}(t(\xi)) = \begin{cases} t_h(\xi)^{\alpha_i+1} & \text{for } \alpha = (0, \dots, 0, \alpha_i, 0, 0, \dots) \\ 0 & \text{otherwise.} \end{cases}$$

2. Steenrod operations. In this section we observe the mod p Steenrod operations in the theory $U_c^*(-)$ and research on them in connection with the Landweber-Novikov operations which are introduced in the previous section.

Let G be a compact Lie group and Z_p a cyclic group of order p with a generator ρ . By a (G, Z_p) -space X we mean a Hausdorff space X having both actions of G and Z_p which commute. Let V be a complex G -module. Throughout this section we only treat finite dimensional complex G -modules. We consider the G -module V a (G, Z_p) -space with a Z_p action defined by $\rho^k v = \exp \frac{2\pi\sqrt{-1}k}{p} v$ ($v \in V$). Then $S(V)^+ = S(V) \cup \{\infty\}$ is a pointed (G, Z_p) -space with a fixed base point ∞ , where $S(V)$ is the unit sphere in V .

Example 1. For a pointed G -space X , the p -fold reduced join $\bigwedge^p X = X \wedge \dots \wedge X$ is a pointed (G, Z_p) -space with a Z_p -action defined by $\rho(x_1 \wedge \dots$

$\wedge x_p) = x_2 \wedge \cdots \wedge x_p \wedge x_1$. We consider the p -fold product $\bigtimes^p X = X \times \cdots \times X$ a (G, Z_p) -space for a G -space X , too.

Example 2. Let $\xi : E \rightarrow X$ be a complex G -vector bundle and denote by $\bigtimes^p \xi$ the p -fold product bundle of ξ . Then the total space $E(\bigtimes^p \xi) = E \times \cdots \times E$ of $\bigtimes^p \xi$ is a (G, Z_p) -space with a Z_p -action defined by $\rho(v_1, \dots, v_p) = (v_2, \dots, v_p, v_1)$.

Let us define a G -space and a pointed G -space as follows :

$$E_v(X) = (S(V) \times X) / Z_p \quad \text{for } (G, Z_p)\text{-space } X,$$

and

$$\tilde{E}_v(X) = (S(V)^+ \wedge X) / Z_p \quad \text{for pointed } (G, Z_p)\text{-space } X.$$

Then we have

Proposition 2.1. *For a (G, Z_p) -space X , there holds*

$$\tilde{E}_v(X^+) = E_v(X)^+.$$

Proposition 2.2. *For a complex G -vector bundle ξ over a compact G -space X*

$$E_v(\bigtimes^p \xi) : E_v(E(\bigtimes^p \xi)) \rightarrow E_v(\bigtimes^p X)$$

is a complex G -vector bundle.

Let $\mathcal{C}(G, Z_p)$ be the category of pointed (G, Z_p) -spaces and $\mathcal{C}(G)$ the category of pointed G -spaces. Then $\tilde{E}_v : \mathcal{C}(G, Z_p) \rightarrow \mathcal{C}(G)$ is a covariant functor.

Furthermore we have

Proposition 2.3. *If ξ is a complex (G, Z_p) -vector bundle over a compact (G, Z_p) -space X , then*

$$E_v(\xi) : E_v(E(\xi)) \rightarrow E_v(X)$$

is a complex G -vector bundle. And, as for the Thom spaces of them, it follows that

$$T(E_v(\xi)) = \tilde{E}_v(T(\xi)).$$

Proposition 2.4. *For a pair (X, A) of a (G, Z_ρ) -space and its subspace, there exist G -homeomorphisms*

$$\tilde{E}_v(X/A) \approx E_v(X)/E_v(A) \approx \tilde{E}_v(X^+)/\tilde{E}_v(A^+).$$

Proposition 2.5. *For a pointed G -space X with the trivial Z_ρ -action and a pointed (G, Z_ρ) -space Y , there exists a G -homeomorphism*

$$\tilde{E}_v(Y \wedge X) \approx \tilde{E}_v(Y) \wedge X.$$

Proposition 2.6. *For a pointed G -space X with the trivial Z_ρ -action and a G -module W , there exists a G -homeomorphism*

$$\tilde{E}_v((\bigwedge^p W^c) \wedge X) \approx T(E_v(\bigotimes^p W) \times X)/T(E_v(\bigotimes^p W) \times *_X),$$

where W^c means the one point compactification of W .

Proof. We have the following G -homeomorphisms

$$\begin{aligned} \tilde{E}_v((\bigwedge^p W^c) \wedge X) &\approx \tilde{E}_v(\bigwedge^p W^c) \wedge X && \text{(by 2.5)} \\ &\approx \tilde{E}_v((\bigotimes^p W)^+) \wedge X \\ &\approx E_v(\bigotimes^p W)^+ \wedge X && \text{(by 2.1)} \\ &\approx T(E_v(\bigotimes^p W)) \wedge X && \text{(by 2.2)} \\ &= T(E_v(\bigotimes^p W)) \wedge (X^+/*_X^+) \\ &= T(E_v(\bigotimes^p W)) \wedge X^+/T(E_v(\bigotimes^p W)) \wedge *_X^+ \\ &= T(E_v(\bigotimes^p W) \times X)/T(E_v(\bigotimes^p W) \times *_X). \quad \text{q.e.d.} \end{aligned}$$

By the same way as in the non-equivariant case we have the following Thom isomorphism theorem of a pair (cf. [2]):

Theorem 2.7. *For an n -dimensional complex G -vector bundle ξ over a compact G -space X and a closed G -subspace A of X , the Thom homomorphism*

$$\phi : U_G^*(X, A) \rightarrow U_G^{*+2n}(T(\xi), T(\xi|A))$$

is an isomorphism.

In virtue of Proposition 2.3, for a G -module W and a pointed G -space

X with the trivial Z_p -action,

$$\bigotimes^p W : E_v((\bigotimes^p W) \times X) \rightarrow E_v((*, \dots, *) \times X)$$

is a G -vector bundle. Therefore, by making use of Theorem 2.7 and Propositions 2.4 and 2.6, we obtain a Thom isomorphism

$$\phi : \tilde{U}_G^*(\tilde{E}_v(X)) \rightarrow \tilde{U}_G^*(\tilde{E}_v((\bigwedge W^c) \wedge X)).$$

We now would like to define the external mod p Steenrod operation

$$P_v^{2k} : \tilde{U}_G^{2k}(X) \rightarrow \tilde{U}_G^{2pk}(E_v(X))$$

for each G -module V and a pointed G -space X .

Let $x \in U_G^{2k}(X)$ be represented by $f : W^c \wedge X \rightarrow M_{1|w_1+k}(U) \subset M_{1|w_1+k}(G)$. Consider the composition of G -maps

$$\begin{aligned} \tilde{E}_v(\bigwedge f) : \tilde{E}_v(\bigwedge (W^c \wedge X)) &\rightarrow \tilde{E}_v(\bigwedge M_{1|w_1+k}(U)) \\ &= \tilde{E}_v(T(\bigotimes^p \gamma_G^{1|w_1+k}(U))) \\ &= T(E_v(\bigotimes^p \gamma_G^{1|w_1+k}(U))) \quad (\text{by 2.3}) \\ &\xrightarrow{\mu_p} T(\gamma_G^{p|1|w_1+k}) = M_{p|1|w_1+k}(G), \end{aligned}$$

where μ_p is the map of Thom spaces induced by the classifying map of the complex G -vector bundle $E_v(\bigotimes^p \gamma_G^{1|w_1+k}(U))$. The map μ_p represents the Thom class

$$[\mu_p] = t(E_v(\bigotimes^p \gamma_G^{1|w_1+k}(U))) \in \tilde{U}_G^{2p|1|w_1+k}(T(E_v(\bigotimes^p \gamma_G^{1|w_1+k}(U)))).$$

Define a map $\tilde{d} : (\bigwedge W^c) \wedge X \rightarrow \bigwedge (W^c \wedge X)$ by $\tilde{d}((w_1 \wedge \dots \wedge w_p) \wedge x) = (w_1 \wedge x) \wedge \dots \wedge (w_p \wedge x)$. Then we get a G -map

$$\tilde{E}_v(\tilde{d}) : \tilde{E}_v((\bigwedge W^c) \wedge X) \rightarrow \tilde{E}_v(\bigwedge (W^c \wedge X)).$$

Now we define $P_v^{2k}(x)$ by

$$P_v^{2k}(x) = \phi^{-1} \tilde{E}_v(\tilde{d}) * \tilde{E}_v(\bigwedge f) * (t(E_v(\bigotimes^p \gamma_G^{1|w_1+k}(U)))).$$

And we have the following properties :

Proposition 2.8. *For a G -module V there exists an operator*

$$P_v : U_G^*(-) \rightarrow U_G^*(E_v(-))$$

with the following properties :

- (1) (naturality) $P_v^k h^*(x) = E_v(h)^* P_v^k(x)$.
- (2) (multiplicativity) For $x \in U_G^{2k}(X)$ and $y \in U_G^{2l}(Y)$

$$P_v^{2k+2l}(x \times y) = P_v^{2k}(x) \times P_v^{2l}(y).$$

- (3) For the Thom class $t(\xi) \in \tilde{U}_G^{2k}(T(\xi))$ of a k -dimensional G -vector bundle ξ ,

$$P_v^k(t(\xi)) = \tilde{E}_v(\tilde{d})^*(t(E_v(\bigotimes^p \xi))) = t(E_v(\xi)).$$

Let L be the canonical 1-dimensional Z_p -module and consider it a trivial G -module. Put

$$\Delta = L \oplus L^2 \oplus \dots \oplus L^{p-1}.$$

Then we obtain

Proposition 2.9. *Let ξ be a complex G -vector bundle over a compact G -space X . Consider the p -fold sum $\xi \oplus \dots \oplus \xi$ a (G, Z_p) -bundle over X with a Z_p -action defined by $\rho(v_1, \dots, v_p) = (v_2, \dots, v_p, v_1)$ for $(v_1, \dots, v_p) \in E(\xi \oplus \dots \oplus \xi)$. Then it follows that*

- (1) the vector bundles $\xi \oplus \dots \oplus \xi$ and $\xi \otimes (C \oplus \Delta)$ are (G, Z_p) -isomorphic, and

- (2) the diagram

$$\begin{array}{ccc} \xi & \xrightarrow{\hat{d}} & \xi \oplus \dots \oplus \xi \\ & \searrow i & \downarrow \cong \\ & & \xi \otimes (C \oplus \Delta) \end{array}$$

is G -homotopy commutative, where \hat{d} is the diagonal map and i is the natural inclusion defined by $i(v) = v \otimes 1 \in \xi \otimes C$.

Proof. (1) Let us consider a (p, p) -matrix A and a unitary matrix $U = (u_{ij})$ such that

$$A = \begin{bmatrix} 0 & \dots & 0 & 1 \\ 1 & 0 & \dots & 0 \\ & 1 & \dots & \vdots \\ & & \dots & 1 & 0 \\ 0 & & & & \end{bmatrix}, \quad U^{-1}AU = \begin{bmatrix} 1 & & & 0 \\ & \rho & & \\ & & \dots & \\ 0 & & & \rho^{p-1} \end{bmatrix}.$$

Then a (G, Z_p) -bundle isomorphism

$$h : \xi \oplus \cdots \oplus \xi \rightarrow \xi \otimes (C \oplus \Delta) = \xi \otimes C \oplus \xi \otimes L \oplus \cdots \oplus \xi \otimes L^{p-1}$$

and its inverse h^{-1} are given by

$$h(v_1, \dots, v_p) = \left(\sum_{j=1}^p u_{j1} v_j \otimes 1, \sum_{j=1}^p u_{j2} v_j \otimes 1, \dots, \sum_{j=1}^p u_{jp} v_j \otimes 1 \right)$$

and

$$h^{-1}(v_1 \otimes z_1, \dots, v_p \otimes z_p) = \left(\sum_{j=1}^p u'_{j1} z_j v_j, \dots, \sum_{j=1}^p u'_{jp} z_j v_j \right)$$

for $(v_1, \dots, v_p) \in \xi \oplus \cdots \oplus \xi$ and $(v_1 \otimes z_1, \dots, v_p \otimes z_p) \in \xi \otimes (C \oplus \Delta)$, where $U^{-1} = (u'_{ij})$.

(2) Since there holds $h\rho = \rho h$ for the generator $\rho \in Z_p$, it follows that

$$\begin{aligned} h\hat{d}(v) &= h\rho(v, \dots, v) = \left(\sum_{j=1}^p u_{j1} v \otimes 1, \dots, \sum_{j=1}^p u_{jp} v \otimes 1 \right) \\ &= \rho \left(\sum_{j=1}^p u_{j1} v \otimes 1, \dots, \sum_{j=1}^p u_{jp} v \otimes 1 \right) \\ &= \left(\sum_{j=1}^p u_{j1} v \otimes 1, \sum_{j=1}^p u_{j2} v \otimes \rho \cdot 1, \dots, \sum_{j=1}^p u_{jp} v \otimes \rho^{p-1} \cdot 1 \right). \end{aligned}$$

Hence we have

$$\left(\sum_{j=1}^p u_{jk} \right) (v \otimes 1) = \rho^{k-1} \left(\sum_{j=1}^p u_{jk} \right) (v \otimes 1) \quad \text{in } \xi \otimes L^{k-1}.$$

This implies

$$\begin{aligned} \sum_{j=1}^p u_{jk} &= 0 \quad (k = 2, \dots, p), \text{ that is,} \\ h\hat{d}(v) &= \left(\sum_{j=1}^p u_{j1} v \otimes 1, 0, \dots, 0 \right). \end{aligned}$$

Since (u_{j1}) is an eigenvector for 1 of A , we have $u_{11} = \cdots = u_{p1}$ and $|u_{11}| = \frac{1}{\sqrt{p}}$. Hence a G -homotopy connecting $h\hat{d}$ and i is given easily. q. e. d.

Proposition 2.10. *For complex G -vector bundles ξ and η over a compact G -space X , let*

$$i_\xi : T(\xi) \rightarrow T(\xi \oplus \eta)$$

be an inclusion given by $i_\xi(v) = (v, 0)$ for $v \in T(\xi)$, and

$$\phi_\xi : U_c^*(X) \rightarrow \tilde{U}_c^*(T(\xi))$$

the Thom isomorphism. Then there holds

$$i_\xi^*(t(\xi \oplus \eta)) = \phi_\xi(e(\eta))$$

for the Thom class $t(\xi \oplus \eta) \in \tilde{U}_c^*(T(\xi \oplus \eta))$ and the Euler class $e(\eta) \in U_c^*(X)$.

Proof. Consider the following commutative diagram

$$\begin{array}{ccccc} \tilde{U}_c^*(T(\xi)) \otimes \tilde{U}_c^*(T(\eta)) & \xrightarrow{\times} & \tilde{U}_c^*(T(\xi) \wedge T(\eta)) & \xrightarrow{\bar{d}^*} & \tilde{U}_c^*(T(\xi \oplus \eta)) \\ \downarrow 1 \otimes s^* & & \downarrow (1 \wedge s)^* & & \downarrow i_\xi^* \\ \tilde{U}_c^*(T(\xi)) \otimes \tilde{U}_c^*(X^+) & \xrightarrow{\times} & \tilde{U}_c^*(T(\xi) \wedge X^+) & \xrightarrow{\bar{d}^*} & \tilde{U}_c^*(T(\xi)), \end{array}$$

where $s : X^+ \rightarrow T(\eta)$ is the 0-section and $\bar{d} : T(\xi \oplus \eta) \rightarrow T(\xi \times \eta) = T(\xi) \wedge T(\eta)$ is the map induced by the diagonal map. Then it follows that

$$\begin{aligned} i_\xi^*(t(\xi \oplus \eta)) &= i_\xi^* \bar{d}^*(t(\xi) \times t(\eta)) \\ &= \bar{d}^*(t(\xi) \times s^*(t(\eta))) \\ &= \bar{d}^*(t(\xi) \times e(\eta)) \\ &= \phi_\xi(e(\eta)). \end{aligned} \quad \text{q.e.d.}$$

Proposition 2.11. For an n -dimensional complex G -vector bundle ξ over a compact G -space X , there holds

$$P_V^{2n}(t(\xi)) = \phi_{E_V, i_\xi}(e(E_V(\xi \otimes \Delta))).$$

Proof. By Proposition 2.8 we have

$$P_V^{2n}(t(\xi)) = \tilde{E}_V(\bar{d})^*(t(E_V(\overset{p}{\times} \xi))).$$

Since the commutative diagram

$$\begin{array}{ccc} T(\xi) & \xrightarrow{\hat{d}} & T(\xi \oplus \dots \oplus \xi) \\ & \searrow \bar{d} & \downarrow \bar{d} \\ & & T(\xi \times \dots \times \xi) \end{array}$$

induces $\tilde{E}_V(\bar{d})^* = \tilde{E}_V(\hat{d})^* \tilde{E}_V(\bar{d})^*$, we get

$$\begin{aligned}
 \tilde{E}_v(\bar{d})^*(t(E_v(\bigotimes^p \xi))) &= \tilde{E}_v(\hat{d})^*(t(E_v(\xi \oplus \dots \oplus \xi))) \\
 &= \tilde{E}_v(i_\xi)^*(t(E_v(\xi \oplus \xi \otimes \Delta))) \quad (\text{by 2.9 (2)}) \\
 &= i_{E_v(\theta)}^*(t(E_v(\xi) \oplus E_v(\xi \otimes \Delta))) \\
 &= \phi_{E_v(\theta)}(e(E_v(\xi \otimes \Delta))) \quad (\text{by 2.10}) \quad \text{q.e.d.}
 \end{aligned}$$

Let us consider a connection of the operations P_v with the Landweber-Novikov operations introduced in § 1. Therefore, let us assume the compact Lie group G abelian hereafter.

Let $V = L_1 \oplus \dots \oplus L_m$ and $W = L'_1 \oplus \dots \oplus L'_n$ be complex G -modules, where L_i and L'_j are 1-dimensional complex G -modules. Let $P(V)$ be the complex projective space for the G -module V and $\eta(V; C)$ the canonical complex G -line bundle over $P(V)$. Then, according to [4, Theorems 4.2 and 4.5] we see that

$$h_c^*(P(V) \times P(W)) = h_c^*(\text{pt})[x_v, y_w] / (\theta_v(x_v), \theta_w(y_w))$$

where $x_v = e_h(\eta(V; C) \hat{\otimes} 1)$ and $y_w = e_h(1 \hat{\otimes} \eta(W; C))$ are the Euler classes of the G -line bundles, and $(\theta_v(x_v), \theta_w(y_w))$ is an ideal generated by polynomials $\theta_v(x_v) = (x_v - e_h(L_1)) \dots (x_v - e_h(L_m))$ and $\theta_w(y_w) = (y_w - e_h(L'_1)) \dots (y_w - e_h(L'_n))$. As usual we put

$$h_c^*(P_\infty \times P_\infty) = \lim h_c^*(P(V) \times P(W))$$

where the limit depends on the inverse system defined by inclusion maps of G -modules. Then we get

$$h_c^*(P_\infty \times P_\infty) = h_c^*(\text{pt})[[x, y]].$$

As usual, by commutativity and associativity of tensor products of G -vector bundles, we obtain a commutative formal group

$$F(x, y) = \sum a_{ij} x^i y^j \in h_c^2(P_\infty \times P_\infty)$$

such that $F(x, y) | P(V) \times P(W) = e_h(\eta(V; C) \hat{\otimes} \eta(W; C))$ and $a_{10} = a_{01} = 1$. And, for G -line bundles ξ and η over a compact G -space X , we have

$$e_h(\xi \otimes \eta) = F(e_h(\xi), e_h(\eta)) = e_h(\xi) + e_h(\eta) + \text{higher terms.}$$

Lemma 2.12. *For an n -dimensional complex G -vector bundle ξ over a compact G -space X , there holds*

$$s_0 P_v^{2n}(t(\xi)) = \sum_{|\alpha| \leq n} e_h(E_v(\Delta))^{n-|\alpha|} b_\alpha(v) s_\alpha(t(\xi))$$

where $v = e_h(E_V(L))$, $|\alpha| = \sum \alpha_i$ for each sequence $\alpha = (\alpha_1, \alpha_2, \dots)$ and $b_\alpha(v) \in h_c^*(\text{pt})[[v]]$ is a power series.

Proof. By Proposition 2.11 we have

$$\begin{aligned} s_0 P_V^{2n}(t(\xi)) &= s_0 \phi_{E_V(\xi)}(e(E_V(\xi \otimes \Delta))) \\ &= \phi(E_V(\xi))(e_h(E_V(\xi \otimes \Delta))). \end{aligned}$$

1) When ξ is a sum of G -line bundles ξ_1, \dots, ξ_n , it follows that

$$\begin{aligned} e_h(E_V(\xi \otimes \Delta)) &= e_h(E_V(\xi_1 \otimes \Delta \oplus \dots \oplus \xi_n \otimes \Delta)) \\ &= e_h(\xi_1 \hat{\otimes} E_V(\Delta) \oplus \dots \oplus \xi_n \hat{\otimes} E_V(\Delta)) \\ &= e_h(\xi_1 \hat{\otimes} E_V(\Delta)) \cdots e_h(\xi_n \hat{\otimes} E_V(\Delta)). \end{aligned}$$

For each k we have

$$\begin{aligned} e_h(\xi_k \hat{\otimes} E_V(\Delta)) &= e_h(\xi_k \hat{\otimes} E_V(L) \oplus \dots \oplus \xi_k \hat{\otimes} E_V(L^{p-1})) \\ &= e_h(\xi_k \hat{\otimes} E_V(L)) \cdots e_h(\xi_k \hat{\otimes} E_V(L^{p-1})) \\ &= F(e_h(\xi_k), e_h(E_V(L))) \cdots F(e_h(\xi_k), e_h(E_V(L^{p-1}))) \\ &= \prod_{l=1}^{p-1} (e_h(E_V(L^l)) + \sum_{j \geq 1} a_j(v) e_h(\xi_k)^j) \\ &= e_h(E_V(\Delta)) + \sum_{j \geq 1} b_j(v) e_h(\xi_k)^j, \end{aligned}$$

where $a_j(v)$ and $b_j(v)$ are formal power series of v . Hence we have

$$\begin{aligned} e_h(E_V(\xi \otimes \Delta)) &= \prod_{k=1}^n (e_h(E_V(\Delta)) + \sum_{j \geq 1} b_j(v) e_h(\xi_k)^j) \\ &= \sum_{|\alpha| \leq n} e_h(E_V(\Delta))^{n-|\alpha|} b_\alpha(v) c_\alpha(\xi) \end{aligned}$$

where $c_\alpha(\xi) = \sum e_h(\xi_1)^{\alpha_1} \cdots e_h(\xi_n)^{\alpha_n}$ and $b_\alpha(v)$ is a formal power series of v . Therefore we have

$$\begin{aligned} s_0 P_V^{2n}(t(\xi)) &= \phi(E_V(\xi)) \left(\sum_{|\alpha| \leq n} e_h(E_V(\Delta))^{n-|\alpha|} b_\alpha(v) c_\alpha(\xi) \right) \\ &= \sum_{|\alpha| \leq n} e_h(E_V(\Delta))^{n-|\alpha|} b_\alpha(v) s_\alpha(\xi). \end{aligned}$$

2) General case is shown by making use of the splitting principle in the theory $h_c^*(-)$. q. e. d.

Now we obtain an h_c^* -theoretic version of [15, Proposition 3.17].

Theorem 2.13. *Let $x \in \tilde{U}_c^{2n}(X)$ be represented by a map $f: W^c \wedge X \rightarrow M_{|W|+n}(U) \subset M_{|W|+n}(G)$. Then there holds*

$$e_h(E_r(\Delta))^{||W||} s_0 P_V^{2n}(x) = \sum_{|\alpha| \leq ||W||+n} e_h(E_r(\Delta))^{n-||W||-|\alpha|} b_\alpha(v) s_\alpha(x),$$

where $b_\alpha(v) \in h_c^*(pt)[[v]]$ is a well defined power series.

Proof. There holds $x = \sigma_w^{-1} f^*(t(\gamma_G^m(U)))$, where $m = ||W|| + n$ and σ_w is the suspension isomorphism. Hence, by the previous lemma, the naturality of s_ω and P_V^{2m} and the stability of s_ω , we have

$$\begin{aligned} s_0 P_V^{2m}(f^*(t(\gamma_G^m(U)))) &= \tilde{E}_V(f)^*(s_0 P_V^{2m}(t(\gamma_G^m(U)))) \\ &= \tilde{E}_V(f)^*(\sum_{|\alpha| \leq m} e_h(E_r(\Delta))^{m-|\alpha|} b_\alpha(v) s_\alpha(t(\gamma_G^m(U)))) \\ &= \sum_{|\alpha| \leq m} e_h(E_r(\Delta))^{m-|\alpha|} b_\alpha(v) s_\alpha(f^*(t(\gamma_G^m(U)))) \\ &= \sum_{|\alpha| \leq m} e_h(E_r(\Delta))^{m-|\alpha|} b_\alpha(v) s_\alpha(\sigma_w x) \\ &= \sum_{|\alpha| \leq m} e_h(E_r(\Delta))^{m-|\alpha|} b_\alpha(v) \sigma_w s_\alpha(x). \end{aligned}$$

On the other hand we have

$$\begin{aligned} s_0 P_V^{2m}(\sigma_w x) &= s_0 P_V^{2m}(\sigma_w(1) \times x) \\ &= s_0 P_V(\sigma_w(1)) \times s_0 P_V(x). \end{aligned}$$

Since $\sigma_w(1) = t(W)$, by the previous lemma, we have

$$s_0 P_V(\sigma_w(1)) = \sum_{|\alpha| \leq ||W||} e_h(E_r(\Delta))^{||W||-|\alpha|} b_\alpha(v) s_\alpha(\sigma_w(1)).$$

Here

$$s_\alpha \sigma_w(1) = \sigma_w s_\alpha(1) = \begin{cases} \sigma_w(1) & \text{for } \alpha = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Thus we have

$$s_0 P_V^{2m}(\sigma_w x) = e_h(E_r(\Delta))^{||W||} \sigma_w s_0 P_V^{2n}(x).$$

This completes the proof.

q. e. d.

Acknowledgement. The author wishes to express his thanks to Professor M. Kamata for his kind advices and discussions.

REFERENCES

- [1] J. F. ADAMS : P. S. Novikov's work on operations on complex cobordism, mimeographed, Univ. of Chicago, 1967.
- [2] T. BRÖCKER und T. tom-DIECK : Kobordismentheorie, Lecture Notes in Math. 178, Springer-

- Verlag, 1970.
- [3] P. E. CONNER and E. E. FLOYD : The relation of cobordism to K-theories, Lecture Notes in Math. 28, Springer-Verlag, 1966.
 - [4] M. FUJII and M. KAMATA : On the completion of the G -equivariant unitary cobordism rings of G -spaces, Publ. R. I. M. S., Kyoto Univ. 19 (1983), 577–600.
 - [5] M. FUJII and S. KÔNO : A note on the local triviality of G -vector bundles, Yokohama Math. J. 30 (1982), 49–52.
 - [6] T. tom-DIECK : Kobordismen-Theorie und Transformations-Gruppen, Preprint Series, Aarhus Univ., (1968/69).
 - [7] T. tom-DIECK : Bordism of G -manifolds and integrality theorems, Topology 9 (1970), 345–358.
 - [8] T. tom-DIECK : Steenrod-Operationen in Kobordismen-Theorien, Math. Z. 107 (1968), 380–401.
 - [9] P. S. LANDWEBER : Cobordism operations and Hopf algebras, Trans. Amer. Math. Soc. 129 (1967), 94–110.
 - [10] M. NAKAOKA : Characteristic Classes with values in complex Cobordism, Osaka J. Math. 10 (1973), 521–543.
 - [11] P. S. NOVIKOV : The method of algebraic topology from viewpoint of cobordism theories (Russian), Izv. Akad. Nauk SSSR 31 (1967), 855–951.
 - [12] C. OKONEK : Der Conner-Floyd-Isomorphismus für Abelsche Gruppen, Math. Z. 179 (1982), 201–212.
 - [13] C. OKONEK : Die äquivariante unitäre Kobordismentheorie und ihre äquivariante formale Gruppe, Arch. Math. 40 (1983), 208–220.
 - [14] D. QUILLEN : On the formal group laws of unoriented and complex cobordism theory, Bull. Amer. Math. Soc. 75 (1969), 1293–1298.
 - [15] D. QUILLEN : Elementary proofs of some results of cobordism theory using Steenrod operations, Advances in Math. 7 (1971), 29–56.

OKAYAMA UNIVERSITY

(Received February 15, 1988)