

BORDISM THEORY WITH REALITY AND DUALITY THEOREM OF POINCARÉ TYPE

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Introduction. In the paper [10] we introduced the cobordism theory with reality. The purpose of this paper is to form a bordism theory of closed differentiable manifolds with a given real structure on the stable tangent bundle.

In this paper, by a τ -space (X, t) we mean a Hausdorff space X together with an involution $t : X \rightarrow X$, and by a τ -map we mean an equivariant map between τ -spaces (which we called a real space and a real map in the previous papers [10], [11]). By a τ -manifold (M, t) we mean a compact C^∞ -manifold M together with a C^∞ -involution $t : M \rightarrow M$.

In § 1 we consider weakly real structures on equivariant vector bundles over τ -spaces. In § 2 we introduce an R -structure on the τ -manifold, which is an equivalence class of the real structures on the stable tangent bundle. The τ -manifold with an R -structure is called an R -manifold. In § 3 we form the bordism groups of R -manifolds and in § 4 we show that there is a duality of Poincaré type between the bordism groups and the cobordism groups with reality.

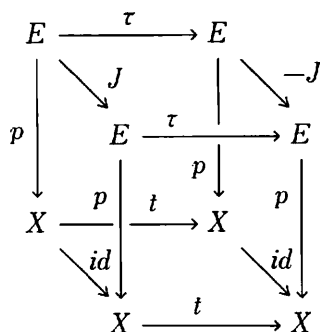
1. Weakly real structures of equivariant vector bundles. By a τ -space (X, t) we mean a Hausdorff space X together with an involution $t = t_x : X \rightarrow X$, and by a τ -map we mean an equivariant map between τ -spaces.

By an *equivariant vector bundle* $\xi = (E, p, X, \tau)$ over a τ -space (X, t) we mean a vector bundle (E, p, X) such that

- i) the total space (E, τ) is a τ -space, and
- ii) the projection $p : E \rightarrow X$ is a τ -map.

Furthermore, by a *real vector bundle* $\xi = (E, p, X, J, \tau)$ over a τ -space (X, t) we mean an equivariant vector bundle (E, p, X, τ) such that

- iii) $J : E \rightarrow E$ is a bundle automorphism such that $J^2 = -id$, $\tau J = (-J)\tau$ and $p(-J) = p$. That is, the diagram



commutes. (E, p, X, J) is a complex vector bundle and the bundle map $\tau: E \rightarrow E$ is an isomorphism from the complex vector bundle (E, J) to the conjugate bundle $(E, -J)$. (J, τ) will be called a *real structure* of the vector bundle $\xi = (E, p, X)$.

Let $\xi_i = (E_i, p_i, X_i, J_i, \tau_i)$ be a real vector bundle over a τ -space (X_i, t_i) ($i = 1$ or 2). Then a *real vector bundle map* $f: \xi_1 \rightarrow \xi_2$ is a bundle map f such that $f\tau_1 = \tau_2f$ and $fJ_1 = J_2f$. If $(X_1, t_1) = (X_2, t_2)$ and f is a bundle isomorphism, then the real bundle map $f: \xi_1 \rightarrow \xi_2$ is said to be an *isomorphism* of real vector bundles. When $(X_1, t_1) = (X_2, t_2)$, we can form a *Whitney sum* of the real vector bundles ξ_1 and ξ_2 by defining

$$\xi_1 \oplus \xi_2 = (E_1 \oplus E_2, p, X_1, J_1 \oplus J_2, \tau_1 \oplus \tau_2).$$

Let R^n be the n -dimensional Euclidean space and let $R^{r,s} = R^r \times R^s$. Consider the *standard involution* $c_{r,s}$ on $R^{r,s}$ defined by

$$c_{r,s}(x_1, \dots, x_r, y_1, \dots, y_s) = (-x_1, \dots, -x_r, y_1, \dots, y_s)$$

for $(x_1, \dots, x_r, y_1, \dots, y_s) \in R^{r,s}$. Furthermore, consider the standard map $I_n: R^{n,n} \rightarrow R^{n,n}$ defined by

$$I_n(x_1, \dots, x_n, y_1, \dots, y_n) = (y_1, \dots, y_n, -x_1, \dots, -x_n)$$

for $(x_1, \dots, x_n, y_1, \dots, y_n) \in R^{n,n}$. Then it holds a relation

$$c_n I_n = -I_n c_n.$$

where $c_n = c_{n,n}$. That is, I_n is the *standard complex structure* on $R^{n,n}$ ($\equiv R^{2n} \equiv C^n$), c_n is the *standard conjugation* on $R^{n,n}$ and (I_n, c_n) is the *standard real structure* on $R^{n,n}$ in the sense of Atiyah [5].

Real vector bundles $\xi = (E, p, X, J, \tau)$ and $\eta = (E', p', X, J', \tau')$ over a τ -space (X, t) are said to be *stably equivalent* if and only if there exist

positive integers m, n and an isomorphism

$$\begin{aligned} \xi \oplus (X \times R^{m,m}, p_1, X, id \times I_m, t \times c_m) \\ \cong \eta \oplus (X \times R^{n,n}, p_1, X, id \times I_n, t \times c_n) \end{aligned}$$

of real vector bundles.

Let $\xi = (E, p, X, \tau)$ be an equivariant vector bundle over a τ -space (X, t) . When there exist positive integers r, s and a real structure $(J, \tau \oplus c_{r,s})$ of the vector bundle $\xi \oplus (X \times R^{r,s}, p_1, X, t \times c_{r,s})$, the real structure $(J, \tau \oplus c_{r,s})$ will be called a *weakly real structure* of the equivariant vector bundle ξ . Two weakly real structures $(J_1, \tau \oplus c_{p,q})$ and $(J_2, \tau \oplus c_{r,s})$ of the equivariant vector bundle ξ are said to be *equivalent* if the real vector bundles $(E \oplus X \times R^{p,q}, p, X, J_1, \tau \oplus c_{p,q})$ and $(E \oplus X \times R^{r,s}, p, X, J_2, \tau \oplus c_{r,s})$ are stably equivalent.

Let $\xi = (E, p, X, \tau)$ and $\eta = (E', p', X, \tau')$ be equivariant vector bundles over a τ -space (X, t) and let $h : \xi \rightarrow \eta$ an isomorphism of equivariant vector bundles. Let $(J, \tau' \oplus c_{r,s})$ be a weakly real structure of η . Then a weakly real structure $(h^*J, \tau \oplus c_{r,s})$ of ξ is induced by the isomorphism

$$\begin{aligned} h \oplus id : \xi \oplus (X \times R^{r,s}, p_1, X, t \times c_{r,s}) \\ \cong \eta \oplus (X \times R^{r,s}, p_1, X, t \times c_{r,s}) \end{aligned}$$

of equivariant vector bundles, where $h^*J = (h \oplus id)^{-1}J(h \oplus id)$.

Often we denote the equivariant vector bundle $(X \times R^{r,s}, p_1, X, t \times c_{r,s})$ simply by $(\underline{R}^{r,s}, c_{r,s})$ or $\underline{R}^{r,s}$, and denote the real vector bundle $(X \times R^{m,m}, p_1, X, id \times I_m, t \times c_m)$ simply by $(\underline{R}^{m,m}, I_m, c_m)$ or $\underline{R}^{m,m}$.

2. R -manifolds. By an m -dimensional τ -manifold (M, t) we mean an m -dimensional compact C^∞ -manifold M together with a C^∞ -involution $t : M \rightarrow M$. As for τ -manifolds we have the following imbedding theorem.

Proposition 2.1. *Any m -dimensional τ -manifold (M, t) without boundary can be equivariantly imbedded as a closed subset of $R^{n,n}$ for some positive integer n .*

This is deduced from a more general situation [6], Theorem 4.1, in the category of C^∞ -manifolds with G -action and equivariant C^∞ -maps.

Furthermore, in virtue of the imbedding theorem, we have the collaring theorem, approximation theorems and the straightening angles in our cate-

gory.

Let (M, t) be a τ -manifold of dimension m and let $\tau(M)$ be its tangent bundle. When there exists a weakly real structure $(J, dt \oplus c_{r,s})$ of the equivariant vector bundle $\tau(M)$, an equivalence class $\Phi = \{(J, dt \oplus c_{r,s})\}$ of the weakly real structure $(J, dt \oplus c_{r,s})$ is called an R -structure of the τ -manifold (M, t) . An R -manifold (M, Φ) is a pair consisting of a τ -manifold (M, t) and an R -structure Φ of (M, t) . Often we denote an R -manifold (M, Φ) simply by M .

Let (M, Φ) be an R -manifold and $(J, dt \oplus c_{r,s})$ a representative of its R -structure Φ . Then $(J \oplus (-I_1), dt \oplus c_{r,s} \oplus c_{1,1})$ is also a weakly real structure of the tangent bundle $\tau(M)$. Now, define $-(M, \Phi)$ by $-\Phi = \{(J \oplus (-I_1), dt \oplus c_{r,s} \oplus c_{1,1})\}$ and $-(M, \Phi) = (M, -\Phi)$.

Let (M, Φ) be an R -manifold with boundary ∂M and $(J, dt \oplus c_{r,s})$ a representative of its R -structure Φ . In virtue of the equivariant collaring theorem, there exists an isomorphism of equivariant vector bundles

$$h : \tau(\partial M) \oplus \underline{R}^{0,1} \rightarrow \tau(M)|_{\partial M},$$

where the positive unit vector of $\underline{R}^{0,1}$ corresponds to an inward unit normal vector in $\tau(M)|_{\partial M}$ by h . Hence $(h^*(J|_{\partial M}), d(t)|_{\partial M}) \oplus c_{r,s+1}$ is a weakly real structure of $\tau(\partial M)$. Let us define $\partial(M, \Phi)$ by $\partial\Phi = \{(h^*(J|_{\partial M}), d(t)|_{\partial M}) \oplus c_{r,s+1}\}$ and $\partial(M, \Phi) = (\partial M, \partial\Phi)$.

Proposition 2.2. *If a τ -manifold (M, t) of dimension m has an R -structure Φ , then the fixed point set F of M by the involution t has a uniform dimension.*

Proof. Let $(J, dt \oplus c_{r,s})$ be a representative of Φ . For any $x \in F$, $((\tau(M) \oplus \underline{R}^{r,s})|_x, J)$ is a complex vector space and $dt \oplus c_{r,s}$ is a conjugate linear involution on it. Therefore the dimension of the fixed point set of it is equal to $m+r+s/2$. Hence $\dim F\tau(M)|_x = m+r-s/2$, where $F\tau(M)|_x$ means the fixed point set of $\tau(M)|_x$. This depends only on the class Φ . q. e. d.

3. Bordism groups of R -manifolds. Let $\mathcal{D}_{p,q}$ denote the family of all $p+q$ dimensional R -manifolds with a weakly real structure of the type $(J, dt \oplus c_{q+r,p+r})$ for some integer r as a representative of the R -structure.

Fix a pair (X, A) consisting of a τ -space X and a τ -subspace A . A (p, q) -singular R -manifold in (X, A) is a triple (M, Φ, f) consisting of

a $p+q$ dimensional R -manifold (M, Φ) in $\mathfrak{D}_{p,q}$ and a τ -map $f : (M, \partial M) \rightarrow (X, A)$. If $A = \phi$, then of course $\partial M = \phi$ also.

A (p, q) -singular R -manifold (M, Φ, f) in (X, A) is said to *bord* if and only if there is a $p+q+1$ dimensional R -manifold (W, Ψ) in $\mathfrak{D}_{p,q+1}$ and a τ -map $F : W \rightarrow X$ for which

- i) M is contained in ∂W as a regular τ -submanifold, and
- ii) $\partial \Psi|_M = \Phi, F|_M = f$ and $F(\partial W - M) \subset A$.

From two (p, q) -singular R -manifolds (M_1, Φ_1, f_1) and (M_2, Φ_2, f_2) a disjoint union $(M_1 \cup M_2, \Phi_1 \cup \Phi_2, f_1 \cup f_2)$ is defined, where $M_1 \cap M_2 = \phi$, $\Phi_1 \cup \Phi_2|_{M_i} = \Phi_i$ and $f_1 \cup f_2|_{M_i} = f_i$. Define $-(M, \Phi, f) = (M, -\Phi, f)$. A pair (M_1, Φ_1, f_1) and (M_2, Φ_2, f_2) of (p, q) -singular R -manifolds in (X, A) are *bordant* if and only if the disjoint union $(M_1 \cup M_2, \Phi_1 \cup -\Phi_2, f_1 \cup f_2)$ *bords* in (X, A) .

Lemma 3.1. *Let (M, Φ) be an R -manifold in $\mathfrak{D}_{p,q}$. If two τ -maps $f, g : (M, \partial M) \rightarrow (X, A)$ are homotopic as τ -maps, then (p, q) -singular R -manifolds (M, Φ, f) and (M, Φ, g) in (X, A) are bordant.*

Proof. There is a τ -homotopy $F : (M \times I, \partial M \times I) \rightarrow (X, A)$ such that $F(x, 0) = f(x)$ and $F(x, 1) = g(x)$ for $x \in M$, where the involution acts trivially on $I = [0, 1]$. Let $p : M \times I \rightarrow M$ be the projection. Then there exists an isomorphism

$$\lambda : \tau(M \times I) \rightarrow p^* \tau(M) \oplus \underline{R}^{0,1}$$

of equivariant vector bundles. Let $(J, dt \oplus c_{r,s})$ be a representative of the R -structure Φ . Then a weakly real structure $(J', d(t \times 1) \oplus c_{r+1,s})$ of $\tau(M \times I)$ is induced by isomorphisms

$$\begin{aligned} \tau(M \times I) \oplus \underline{R}^{r+1,s} &\xrightarrow{\lambda \oplus 1} p^* \tau(M) \oplus \underline{R}^{0,1} \oplus \underline{R}^{r+1,s} \\ &\xrightarrow{\alpha} p^* \tau(M) \oplus \underline{R}^{r,s} \oplus \underline{R}^{1,1} \end{aligned}$$

where $J' = (\lambda \oplus 1)^{-1} \alpha^{-1} (p^* J \oplus I_1) \alpha (\lambda \oplus 1)$. Now, let us define $\Psi = \{(J', d(t \times 1) \oplus c_{r+1,s})\}$. Then we may show that

- 1) $M \cup M$ is a regular τ -submanifold of $\partial(M \times I)$,
- 2) $F(\partial(M \times I) - M \times 0 \cup M \times 1) \subset A$,
- 3) $F|_{M \times 0 \cup M \times 1} = f \cup g$, and
- 4) $\partial \Psi|_{M \times 0} = \Phi$ and $\partial \Psi|_{M \times 1} = -\Phi$.

This shows that (M, Φ, f) and (M, Φ, g) are bordant.

q. e. d.

By a parallel argument to the complex bordism theory we may show that the bordism relation is an equivalence relation. Denote the *bordism class* of (M, Φ, f) by $[M, \Phi, f]$, and the collection of all such bordism classes by $\mathfrak{MR}_{p,q}(X, A)$. Then $\mathfrak{MR}_{p,q}(X, A)$ is also an abelian group by the disjoint union. Given a τ -map $\psi: (X, A) \rightarrow (Y, B)$, there is associated a natural homomorphism $\psi_*: \mathfrak{MR}_{p,q}(X, A) \rightarrow \mathfrak{MR}_{p,q}(Y, B)$ given by $\psi_*([M, \Phi, f]) = [M, \Phi, \psi f]$. There is also a homomorphism $\partial: \mathfrak{MR}_{p,q}(X, A) \rightarrow \mathfrak{MR}_{p,q-1}(A)$ given by $\partial([M, \Phi, f]) = [\partial M, \partial \Phi, f|_{\partial M}]$. Hence we have

Theorem 1. *For any integer p , $\mathfrak{MR}_{p,*}(\ , \)$ is a generalized homology theory on the category of pairs of τ -spaces and τ -maps of pairs.*

4. Poincaré duality. Let (M, Φ) , (N, Ψ) be R -manifolds and $(J_M, dt_M \oplus c_{p,q})$, $(J_N, dt_N \oplus c_{r,s})$ representatives of Φ , Ψ respectively. Let

$$f: M \rightarrow N$$

be a τ -embedding. A real structure (J, e) of the normal bundle $\nu(f)$ is called a *proper real structure* if and only if two real vector bundles $(\nu(f), J, e) \oplus (\tau(M) \oplus \underline{R}^{p,q}, J_M, dt_M \oplus c_{p,q})$ and $(f^* \tau(N) \oplus \underline{R}^{r,s}, J_N, dt_N \oplus c_{r,s})$ are stably equivalent.

Proposition 4.1. *Let (M, Φ) , (N, Ψ) be R -manifolds and $f: M \rightarrow N$ a τ -embedding. If there is a proper real structure of $\nu(f)$, then the real structure is unique up to stable equivalence.*

Proof. Let (J_1, e_1) and (J_2, e_2) be two proper real structures of $\nu(f)$. Then there exist representatives $(J_M, dt_M \oplus c_{p,q})$ and $(J_N, dt_N \oplus c_{r,s})$ of Φ and Ψ respectively, such that

$$\begin{aligned} &(\nu(f), J_1, e_1) \oplus (\tau(M) \oplus \underline{R}^{p,q}, J_M, dt_M \oplus c_{p,q}) \\ &\cong (f^* \tau(N) \oplus \underline{R}^{r,s}, J_N, dt_N \oplus c_{r,s}) \\ &(\nu(f), J_2, e_2) \oplus (\tau(M) \oplus \underline{R}^{p,q}, J_M, dt_M \oplus c_{p,q}) \\ &\cong (f^* \tau(N) \oplus \underline{R}^{r,s}, J_N, dt_N \oplus c_{r,s}) \end{aligned}$$

as real vector bundles. Since M is compact, there exist a real vector bundle ξ over M and a positive integer k satisfying

$$(\tau(M) \oplus \underline{R}^{p,q}, J_M, dt_M \oplus c_{p,q}) \oplus \xi \cong (\underline{R}^{k,k}, I_k, c_k)$$

as real vector bundles (cf. [10], Prop. 1.4, 1.7). Hence we have

$$(\nu(f) \oplus \underline{R}^{k,k}, J_1 \oplus I_k, e_1 \oplus c_k) \cong (\nu(f) \oplus \underline{R}^{k,k}, J_2 \oplus I_k, e_2 \oplus c_k)$$

as real vector bundles. q. e. d.

Proposition 4.2. *Let (M, Φ) and (N, Ψ) be R -manifolds in $\mathfrak{D}_{p,q}$ and $\mathfrak{D}_{m,n}$ respectively, and let $f: M \rightarrow N$ be a τ -imbedding. If $m-n = p-q$, then there is a positive integer k such that, for a τ -imbedding*

$$f': M \rightarrow R^{k,k} \times N$$

defined by $f'(x) = (0, f(x))$ for $x \in M$, the normal bundle $\nu(f')$ has a proper real structure.

Proof. Since $m-n = p-q$ by the assumption, we may choose $(J_M, dt_M \oplus c_{q+s, p+s})$ and $(J_N, dt_N \oplus c_{n+r, m+r})$ satisfying $m+r = p+s$ and $n+r = q+s$ as representatives of Φ and Ψ respectively. In virtue of [10], Propositions 1.4 and 1.7, there exist a real vector bundle $\xi = (E(\xi), J_\xi, e_\xi)$ over M and a positive integer k such that

$$\begin{aligned} (E(\xi), J_\xi, e_\xi) \oplus (\tau(M) \oplus \underline{R}^{q+s, p+s}, J_M, dt_M \oplus c_{q+s, p+s}) \\ \cong (\underline{R}^{k,k}, I_k, c_k) \end{aligned}$$

as real vector bundles. Then we have isomorphisms

$$\begin{aligned} \nu(f') &\cong \underline{R}^{k,k} \oplus \nu(f) \\ &\cong E(\xi) \oplus \tau(M) \oplus \underline{R}^{q+s, p-s} \oplus \nu(f) \\ &\cong E(\xi) \oplus \nu(f) \oplus \tau(M) \oplus \underline{R}^{q+s, p+s} \\ &\cong E(\xi) \oplus f^* \tau(N) \oplus \underline{R}^{n+r, m+r} \end{aligned}$$

of equivariant vector bundles. Hence a real structure of $\nu(f')$ is induced by the above isomorphisms from the real structure $(J_\xi \oplus J_N, e_\xi \oplus dt_N \oplus c_{n+r, m+r})$ of $E(\xi) \oplus f^* \tau(N) \oplus \underline{R}^{n+r, m+r}$. This real structure of $\nu(f')$ is a proper real structure and the proposition follows. q. e. d.

Let (X, Ψ) be an R -manifold without boundary and belong to $\mathfrak{D}_{m,n}$. Let $MR^{\tau, s}(X)$ be the real cobordism group introduced in [10]. A natural duality homomorphism

$$D_R: \mathfrak{MR}_{p,q}(X) \rightarrow MR^{m-p, n-q}(X)$$

is defined by the Pontrjagin-Thom construction as follows: Let $[M, \Phi, f]$ be any element in $\mathfrak{MR}_{p,q}(X)$. We may assume that $f: M \rightarrow X$ is a C^∞ - τ -

map. In virtue of Proposition 2.1, there exists a τ -imbedding

$$h : M \rightarrow R^{l,l}.$$

Then, defining a τ -map

$$f' : M \rightarrow R^{l,l} \times X$$

by $f'(x) = (h(x), f(x))$ for $x \in M$, f' is also a τ -imbedding.

Let us assume $m-n = p-q$ ($m-p = n-q$). Then, by Proposition 4.2, there exists a positive integer k such that, for a τ -imbedding

$$f'' : M \rightarrow R^{k,k} \times R^{l,l} \times X$$

defined by $f''(x) = (0, h(x), f(x))$ for $x \in M$, the normal bundle $\nu(f'')$ has a proper real structure. Since the dimension of the real vector bundle $\nu(f'')$ is $k+l+n-q$, there are a real bundle map

$$g : \nu(f'') \rightarrow \gamma^{k+l+n-q}$$

and an induced map $T(\nu(f'')) \rightarrow MU(k+l+n-q)$ of real Thom spaces (cf. [10], Proposition 1.5). There is then the composite map $d(f)$ given by

$$\begin{aligned} \sum^{k+l,k+l} \wedge X^+ &\rightarrow R^{k,k} \times R^{l,l} \times X / R^{k,k} \times R^{l,l} \times X - \text{Int } D(\nu(f'')) \\ &\rightarrow T(\nu(f'')) \rightarrow MU(k+l+n-q), \end{aligned}$$

and $d(f)$ represents an element of $MR^{m-\rho, n-q}(X)$, where $D(\nu(f''))$ is the total space of the associated unit disk bundle of $\nu(f'')$. The correspondence $[M, \Phi, f] \rightarrow |d(f)|$ is well defined and a homomorphism as usual.

In the case of $m-n \neq p-q$, we may choose two positive integers a and b such that $p-q = (m+a) - (n+b)$. Therefore, considering a τ -map

$$\tilde{f} : M \rightarrow R^{a,b} \times X$$

defined by $\tilde{f}(x) = (0, f(x))$ for $x \in M$ instead of f , we can obtain a duality homomorphism

$$D_R : \mathfrak{MR}_{\rho,q}(X) \rightarrow \tilde{M}R^{m+a-\rho, n+b-q}(\sum^{a,b} \wedge X^+) = MR^{m-\rho, n-q}(X).$$

Generally the above homomorphism D_R is not isomorphic, however we have the following theorem deduced from a general situation [13], lemma 3.2, on the equivariant transversal regularity.

Theorem 2. *For any R -manifold (X, Ψ) in $\mathfrak{D}_{m,n}$ without boundary, the duality homomorphism*

$$D_R : \mathfrak{MR}_{p,q}(X) \rightarrow MR^{m-p,n-q}(X)$$

is an isomorphism whenever $p > q$ and is an epimorphism whenever $p = q$.

Let $MU_*()$ and $MU^*()$ be the complex bordism theory and cobordism theory, respectively. In these theories there exists the duality isomorphism of Atiyah-Poincaré type

$$D : MU_\tau(X) \rightarrow MU^{s-\tau}(X)$$

for any compact weakly complex s -manifold X without boundary.

Let

$$\begin{aligned} \rho_* : \mathfrak{MR}_{p,q}(X) &\rightarrow MU_{p+q}(X) \\ \rho^* : MR^{m,n}(X) &\rightarrow MU^{m+n}(X) \end{aligned}$$

be the natural homomorphisms obtained by ignoring the involutions. Then, for any R -manifold (X, Ψ) in $\mathfrak{D}_{m,n}$ without boundary, we have the following commutative diagram

$$\begin{array}{ccc} \mathfrak{MR}_{p,q}(X) & \xrightarrow{D_R} & MR^{m-p,n-q}(X) \\ \rho_* \downarrow & & \rho^* \downarrow \\ MU_{p+q}(X) & \xrightarrow{D} & MU^{m+n-p-q}(X). \end{array}$$

Therefore, $\ker D_R$ is contained in $\ker \rho_*$. Especially, in the case of $X = \text{pt}$ and $p = q \geq 0$, we have

$$\ker D_R = \ker \rho_*,$$

because $\rho^* : MR^{-p,-p}(\text{pt}) \rightarrow MU^{-2p}(\text{pt})$ is isomorphic (cf. [2], Theorem 4.6). Hence we have

Proposition 4.4. *For any integer $p \geq 0$, isomorphisms*

$$\begin{aligned} \bar{D}_R : \mathfrak{MR}_{p,p}(\text{pt})/\ker \rho_* &\rightarrow MR^{-p,-p}(\text{pt}) \\ \bar{\rho}_* : \mathfrak{MR}_{p,p}(\text{pt})/\ker \rho_* &\rightarrow MU_{2p}(\text{pt}) \end{aligned}$$

are induced by D_R and ρ_* , respectively.

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