

## FQF-3 RINGS

Dedicated to Professor Hisao Tominaga on his 60 th birthday

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### Introduction

In this paper we will introduce a new class of rings which seem to deserve study. In the early 1940's Nakayama introduced and began the study of Frobenius and Quasi-Frobenius algebras, or as it turns out self-injective Noetherian rings. In 1948 Thrall introduced various generalizations of Quasi-Frobenius algebras which he called QF-1, QF-2, and QF-3 algebras. Left QF-3 rings are those rings which have a unique minimal faithful left module; unique and minimal in the following sense: If  $M$  is any left module over the ring which is faithful, then  $M$  has a direct summand isomorphic to this minimal faithful module. These rings have been studied rather extensively [C & R<sub>1</sub>, C & R<sub>2</sub>, T<sub>1</sub>, T<sub>2</sub>, T<sub>3</sub>] with the principal reference being that of Tachikawa [T<sub>2</sub>]. If  $R$  is a left QF-3 ring, it is known that the minimal faithful left module is isomorphic to the direct sum of the injective hulls of a finite number of simple left  $R$ -modules and is also projective.

Another class of rings related to the QF rings is the class of left PF (right PF) rings. These rings have the property that every faithful left (right) module is a generator of the category of left (right) modules. All the PF rings are QF-3 on the appropriate side as follows by the theorems of Osofsky and Utumi [O<sub>1</sub>, U<sub>1</sub>] which show that a left PF ring is a left self-injective ring with a finitely generated essential socle. However, not all QF-3 rings are PF. There is a natural generalization of PF called FPF. A ring  $R$  is called left (right) FPF if every finitely generated faithful left (right) module is a generator. These have been studied in [F & P, P<sub>1</sub>, P<sub>2</sub>, P<sub>3</sub>]. Now, as it turns out for a left FPF semiperfect ring, there is minimal faithful left module  $P$  in the following sense, see [F & P]. If  $M$  is finitely generated and faithful,  $M$  has a direct summand isomorphic to  $P$ . The module  $P$  is necessarily projective but need not be injective nor is  $P$  a summand of every faithful left module. The connection with QF-3 rings is even stronger. If the ring  $R$  is Noetherian, FPF on both sides and semiperfect, then  $R$  is a product of Quasi-Frobenius rings and valuation rings. Faticoni [Fa] showed that all Noetherian FPF rings are products

of Quasi-Frobenius rings and bounded Dedekind prime rings. We will give another proof of this here which suggests there is a radical-like property in the background. We also introduce left (right) FQF-3 rings defined as follows: A ring  $R$  is left FQF-3 if it has a left (right) minimal faithful module  $P$  in the following sense: Every finitely generated faithful left (right) module over  $R$  has a direct summand isomorphic to  $P$ . It is worth noting that the integers are FQF-3 and FPF but not QF-3. Also left FQF-3 does not imply FPF nor does FPF imply FQF-3. To see this one only need note that not all QF-3 rings are FPF and not all Dedekind commutative rings are principal ideal rings. We will show that all domains that are left FQF-3 are semi-principal left ideal domains. Since the class of semiprime FQF-3 rings is larger than the class of semiprime QF-3 rings and has many of the properties of semiprime FPF rings, we will concentrate on this class in this paper.

### Notations and conventions

Throughout  $R$  will denote an associative ring with identity and all modules will be unitary. Any unordained adjective will mean two sided where appropriate (i.e. FPF ring means two sided FPF ring etc.). For a left or right  $R$ -module  $M$  and subset  $N$  we will let  $N^\perp = \{r \in R: nr = 0 \text{ for all } n \in N\}$  and  ${}^\perp N$  will be the corresponding left annihilator of the subset  $N$ . Let  $Z(M)$  denote the singular submodule of  $M$  and  $E(M)$  the injective hull of  $M$ . We will denote the maximal left ring of quotients of  $R$  by  $Q(R)$  and  $Z_l(R)$  denotes the left singular ideal of  $R$  where as  $Z_r(R)$  denotes the right singular ideal of  $R$ . For an  $R$ -module  $M$ ,  $M^n$  stands for the  $n$ -fold direct sum of copies of  $M$ .

**Definition 1.** A ring  $R$  is called left FPF if for every finitely generated left  $R$ -module  $M$  such that  ${}^\perp M = 0$ , there exists an integer  $n > 0$  such that  $R \oplus Y = M^n$ . That is to say,  $M$  is a generator of the category of left  $R$ -modules. There is the obvious notion of a right FPF ring.

**Definition 2.** A ring is left bounded if every essential left ideal contains a non-zero two sided ideal.

**Definition 3.** A ring  $R$  is called left QF-3 if there exists a left  $R$ -module  $M$  such that  ${}^\perp M = 0$  and  $M$  is isomorphic to a direct summand of

every faithful  $R$ -module. (A module is faithful if its annihilator is zero.)

**Definition 4.** A ring  $R$  is called left FQF-3 if there exists a left  $R$ -module  $M$  which is faithful and is isomorphic to a direct summand of every finitely generated faithful left  $R$ -module.

**Remark.** We will refer to the module  $M$  in definitions 2 and 3 as a minimal faithful module. Notice that the minimal faithful module referred to in the last two definitions a priori need not be unique up to isomorphism. It is known that the  $M$  in Definition 2 is unique and is the injective hull of a finite sum of simple  $R$ -modules. In Definition 3 the minimal faithful module is unique in the case when  $R$  is semiprime.

**Lemma 5.** *A left FQF-3 ring is left bounded.*

*Proof.* If  $E$  is an essential left ideal of a left FQF-3 ring, then  $R/E$  can not be faithful for it is singular.

**Lemma 6.** *Let  $R$  be a left FQF-3 ring. Then there is an idempotent  $e$  such that  $Re$  is a minimal faithful left  $R$ -module. Also, all minimal faithful  $R$ -modules are projective and cyclic.*

*Proof.* Suffice it to say that  $R$  is obviously faithful.

**Corollary 6.1.** *If  $R$  is a left FQF-3 ring which is semiperfect, then the minimal faithful modules are all isomorphic.*

*Proof.* Since each minimal faithful is a cyclic projective and each is a direct summand of the other, one applies the Krull-Schmidt theorem on direct sums of indecomposables to obtain the desired conclusion.

In the next section we will see that Corollary 6.1 also holds for semiprime FQF-3 rings.

### Semiprime FQF-3 rings

In [F & P] it is shown that for left FPF rings all semiprime left FPF rings have  $Z_l(R) = Z_r(R) = 0$  and conversely. This does not hold in FQF-3 rings. We do have the following:

**Proposition 7.** *If  $R$  is a semiprime left FQF-3 ring, then  $Z_l(R) = 0$ .*

*Proof.* Let  $Re$  be a minimal faithful left ideal. Let  $x \in Z_l(R)$ . Now for a semiprime ring the ideal  $A$  maximal with respect to  $A \cap Rx = 0$  is  ${}^{\perp}Rx = RxR^{\perp} = xR^{\perp}$ . Also, the ideal  $B = A^{\perp} = {}^{\perp}A$  is an essential extension of  $RxR$  as a left ideal. Since  $R$  is semiprime  $A \cap B = 0$ , and it follows that  $Rx \oplus R/B$  is faithful. We must have a map  $f: (Rx) + R/B \rightarrow Re \rightarrow 0$  which splits. This gives  $\text{Im}(Rx) + \text{Im}(R/B) = Re$ . Now the trace of  $R/B$  in  $R$  is  $A$ , since  $A = {}^{\perp}B$ , so  $(\text{Im}(Rx) + A) \cap Re = Re$ . If  $\text{Im}(Rx) \cap A \cap Re = 0$ ,  $\text{Im}(Rx)$  contains a submodule isomorphic to  $Re$  which would be projective, cyclic and singular which is impossible. So  $\text{Im}(Rx) \cap A \neq 0$ . This yields a non-zero square zero left ideal, another contradiction. Finally,  $\text{Im}(Rx) = 0$  implies  $A \supset Re$  and  $B = 0$ , which leaves only the possibility that  $Rx$  is zero.

At this point we know that for a semiprime left FQF-3 ring that the maximal ring of quotients is a left self-injective von-Neumann regular ring. Is the maximal left quotient ring left FQF-3, too? The answer is provided in the following.

**Proposition 7.1.** *Let  $R$  be a semiprime left FQF-3 ring with maximal left quotient ring  $Q(R)$ . Then  $Q(R)$  is left FQF-3.*

In order to prove this theorem we need some lemmas.

**Lemma 7.2.** *Let  $R$  be a semiprime left FQF-3 ring with minimal faithful left ideal  $Re$ . Then  $eRe = \text{End}(Re)$  is a left FPF ring.*

*Proof.* The trace of  $Re$  in  $R$  is  $ReR$  and  $ReR^{\perp} = 0$  since  ${}^{\perp}Re = 0$ . So  $Re$  is a distinguished finitely generated projective. Since  $Re$  is isomorphic to a direct summand of any module which finitely cogenerates  $Re$ ,  $Re$  is an FPF module and we can apply [Thm 4, P<sub>3</sub>].

**Lemma 7.3.** *Let  $R$  be as above in 7.2. Then for  $Q = Q(R)$ ,  $eQe$  is a left self-injective regular ring FPF ring.*

*Proof.* Since  $Q$  is a left self-injective ring and  $Qe$  is injective and non-singular over  $Q$ ,  $eQe$  is a left self-injective regular ring by Osofsky [O<sub>2</sub>]. We claim  $eQe$  is an  $eRe$  essential extension of  $eRe$  and is therefore the left maximal ring of quotients of  $eRe$ . We can apply [Thm 3.5, F & P] to conclude that  $eQe$  is FPF. To see that  $eQe$  is an essential extension of  $eRe$ , if  $eqe \in eQe$ , there is a essential left ideal  $E$  in  $R$  such that

$Eeqe \subset R$ . Now, if  $eEeqe = 0$  we have  $(ReR)Eeqe = 0$ , so that  $Eeqe = 0$  since  $ReR$  is right faithful. Also, since  $eRe$  is FPF and semiprime, as is easily seen,  $Z_l(eRe) = 0$  by [Thm 2.3, F & P], and  $eQe$  is the maximal left quotient ring of  $eRe$ . Next we have that the width, as defined in [F & P] as the supremum of the  $n$  such that there exists an  $M \neq 0$  and a direct sum  $M^n$  contained in  $eQe$ , is finite, again by the results of [Chapt. 5, F & P]. This implies that the width of  $Re$  is finite for the width of  $eRe$  is the same as the width of  $eQe$  by the results of [Chapt. 5, F & P].

**Corollary 7.4.** *If  $R$  is a semiprime left FQF-3 ring, the minimal faithful modules are all isomorphic and of finite width.*

*Proof.* Let  $Re$  be a minimal faithful left ideal with  $e^2 = e$ . Let  $M$  be any other minimal faithful. Now  $Re \cong M \oplus Y$  for some  $Y$  and  $M \cong Re \oplus W$  for some  $W$ . We have then  $Re \cong M \oplus Y \cong Re \oplus Y \oplus W \cong M \oplus Y \oplus Y \oplus W \cong \dots$  which contradicts the fact that width of  $Re$  is finite, unless  $Y$  or  $W$  is zero.

Using the types of self-injective regular ring as in Goodearl and Boyle [G & B] we next show that the purely infinite part of  $Q$  is zero.

**Lemma 7.5.** *If  $Q$  is type II or type III and the maximal quotient ring of a semiprime left FQF-3 ring, then  $Q$  is zero.*

*Proof.* Type II or III regular left self-injective rings have all direct summands of unbounded width. We are left with  $Q$  is a type  $I_f$  ring.

**Theorem 8.** *A regular left self-injective ring  $Q$  is left FQF-3 iff  $Q$  is of type  $I_f$ .*

*Proof.* The previous results imply that if  $Q$  is FQF-3,  $Q$  is type  $I_f$ . To obtain the implication in the other direction we can first write  $Q = \prod Q_n$  where each  $Q_n$  is FPF. We claim each  $Q_n$  is left FQF-3. Now each  $Q_n$  is isomorphic to the  $n \times n$  matrix ring over a self-injective ring  $R$  of width one, i.e. all idempotents in  $R$  are in the center of  $R$ . Also, all one sided ideals of  $R$  are two sided. We first show  $R$  is left FQF-3. To see this suppose  $M$  is a finitely generated faithful left  $R$ -module. Then we have, since  $R$  is FPF,  $M^l \rightarrow R \rightarrow 0$  exact. This yields  $0 \rightarrow R \rightarrow M^l$  exact. We wish to find an element  $m$  in  $M$  so that  ${}^l m = 0$ . This will give an embedding of  $R$  into  $M$  which will split since  $R$  is injective, and hence that  $R$  is the minimal faithful. To find the element in question we

first reduce to the case where  $Z(M) = 0$ . This can be done since in an FPF ring  $M/Z(M)$  is faithful iff  $M$  is by [F & P]. Since  $M$  is finitely generated, we induce on the number of generators of  $M$ . Since left ideals are two sided, if  $M$  is cyclic any generator of  $M$  will do. If  $M$  is generated by  $m_1, \dots, m_n$ , then since  ${}^+m_1$  is a closed ideal of  $R$  it is a direct summand and generated by a central idempotent  $f$  by [Thm. 2.1A, F & P]. We now take the ring  $Rf$  and the module  $fM$  which is generated by  $k-1$  or fewer elements and apply the induction hypothesis to obtain an element  $b$  of the desired type for the ring  $Rf$ . Then  $m_1 + b$  gives the desired element of  $M$  for the ring  $R$ . Next we show that with  $R$  as above each  $Q_n$  is FQF-3. To establish this note that any f.g. faithful  $Q_n$  module is a finitely generated faithful  $R$  module. Now let  $q$  be the  $n \times n$  matrix over  $R$  with one in the upper left corner. Then we have  $Q_n q Q_n = Q_n$ , as well as  $Rq = R$ . So any faithful  $Q_n$ -module contains a copy of  $R$  hence a copy of  $Rq$  hence a copy  $Q_n q$ . Since  $Q_n q$  is injective this copy must be a direct summand and we have our minimal faithful module.

Finally we need that the product of the  $Q_n$ 's is left FQF-3. The natural candidate for the minimal faithful module is the product of the minimal faithful modules constructed above, but it is not finitely generated. Instead we use  $Q(\prod q_n)$  where the  $q_n$  are the corner elements used in the previous paragraph. Let  $M$  be a finitely generated faithful  $Q$  module. For each  $n$  there is in  $M$  an element  $m_n$  so that  $Qq_n m_n \cong Qq_n$ . Then  $\prod q_n Qq_n = T$  is a self-injective ring of width one, so is FPF. Let  $Q^i \rightarrow M \rightarrow 0$  and  $m_1, \dots, m_n$  be the generators of  $M$  which are the images of the identity of  $Q$  given by the 1 canonical embeddings of  $Q$  into  $Q^i$ . Now take  $N$  to be the  $T$  module generated by  $\{m_1, \dots, m_n\}$ . We claim  $N$  is faithful. If not, there is an ideal  $A$  in  $T$  which annihilates  $N$ . But  $A$  generates an ideal in  $Q$  which annihilates  $QN = M$ . To see this we note that the ideal  $A$  generates in  $Q$  is the product of matrices with entries in  $A$ . It must be that  $N$  is a faithful  $R$ -module. Now, as we have seen we can embed  $T$  in  $N$  and this embedding extends to an embedding of  $Q(\prod q_n)$  into  $M$ . But  $Q(\prod q_n)$  is injective so the embedding splits and we have our minimal faithful.

**Corollary 8.1.** *If  $R$  is left FQF-3, semiprime and has no infinite sets of orthogonal idempotents, or has acc on annihilators, then  $R$  is an order in a semi-simple ring.*

*Proof.* In this case we have that  $Q(R)$  is semi-simple.

**Corollary 8.2.** *If  $R$  is left FQF-3 and simple,  $R$  is simple Artinian.*

*Proof.* All semiprime left FQF-3 are left bounded hence simple ones have no non-essential left ideals and are therefore as stated.

**Proposition 8.3.** *If  $R$  is a semiprime left FQF-3 ring, then the maximal left and right quotient rings of  $R$  are the same and  $R$  is also right non-singular.*

*Proof.* Let  $q$  be any element of  $Q$ . We wish to show that  $qR \cap Q \neq 0$ . Let  $f$  be an idempotent so that  $Qf = Qq$ . Then  $Rq$  is essential in  $B = Rf + Rq$ . We want a non-zero map of  $B$  into  $R$ . Let  $A = {}^+Rq$ , a closed two sided ideal of  $R$ . Also,  $A = {}^+B$ , since  $Rq$  is essential in  $B$ . Now we also have  $B \cap A = 0$ , since  $R \cap B \cap A = 0$ . Take  $C = B \oplus R/R \cap B$ . The module  $C$  is faithful. So  $C = Re \oplus Y$ . Under the canonical map  $A$  embeds in  $R/R \cap B$  as an essential submodule. Since  $A$  is closed the trace of  $R/R \cap B$  is  $A$ . As we have seen before  $Re = Re \cap A + (\text{Tr}(R/A))$ , where  $\text{Tr}(R/A)$  is the trace of  $R/A$  in  $Re$ . We have  $BA = 0$  since  $A$  is two sided, so it follows that the trace of  $R/A$  in  $Re$  is not zero. Since  $B$  embeds under the canonical map as an essential submodule of  $R/A$ , the trace of  $B$  in  $R$  is not zero. A map  $g: B \rightarrow R$  is given by  $g(f) = fz$  and  $g(q) = qx$ , where  $z$  and  $x$  are in  $Q$ . There is an essential ideal  $E$  such that  $E(qfz - qx) = 0$  since  $Rq$  is essential in  $B$ . This says that  $qfz = qx$  is in  $R$ . Finally,  $fz$  is in  $R$  completing the proof.

Focusing on prime rings we have:

**Proposition 9.** *A ring  $R$  is a prime left FQF-3 ring iff  $R$  is Morita equivalent to a bounded semi-principal left ideal domain.*

*Proof.* The property of being FQF-3 is clearly a Morita invariant and if  $R$  is a prime FQF-3 ring then it is a standard part of the Goldie theorems that the endomorphism of the minimal faithful  $Re$  is a domain. Now  $Re$  is a generator, for  $R = Re \oplus R(1-e)$  but since  $R(1-e)$  is faithful  $R(1-e) = X_1 \oplus W_1$  with  $X_1 \cong Re$ . Now if  $W_1 \neq 0$ ,  $W_1$  is f.g. and faithful so  $W_1 = X_2 \oplus W_2$  with  $X_2 \cong Re$ . The ring  $R$  is left Goldie, so has finite Goldie dimension, hence and for some  $n$ ,  $W_n = 0$ , and  $Re$  is a generator. Moreover, a domain that is a left FQF-3 clearly has all finitely generated left ideals principal and is left bounded. To see the converse, if  $R$  is a

bounded domain and  $M$  is a faithful finitely generated module, then  $M/Z(M)$  is also faithful. Next, all semi-principal left ideal domains are Goldie. It follows that  $Q(R)$  is a flat epimorphism and we can apply Goodearl [5.18, G] to conclude that all finitely generated non-singular modules are projective. Since the image of a finitely generated module is finitely generated, the image of any map of  $M/Z(M)$  to  $R$  is cyclic and the result follows.

The next theorem shows that certain factor rings of FQF-3 rings of interest are FQF-3.

**Theorem 10.** *Let  $R$  be a left FQF-3 ring and  $A$  the largest left essential extension of  $Z_l(R)$ . Then  $R/A$  is a left FQF-3 ring.*

*Proof.* With  $A$  as in the hypothesis  $A$  is a closed two sided ideal since  $A$  contains  $Z_l$ . In fact  $A = \{x \in R : Ex \subset Z_l(A) \text{ for some essential left ideal } E \text{ of } R\}$ . If  $A$  is essential, then  $A$  is  $R$  and there is nothing to prove. If  $A$  is not essential, there is a non-zero left ideal  $B$  maximal with respect to  $B \cap A = 0$ . Let  $M$  be a faithful finitely generated  $R/A$  module. Take  $M \oplus R/B = N$ . Then  $N$  is faithful and so  $N$  has a direct summand isomorphic to  $Re$ , where  $Re$  is a minimal faithful left ideal of  $R$ . The ideal  $A$  is invariant in  $R$  and canonically embeds in  $R/B$  as an essential submodule. As in the proof of 8.3 the trace of  $R/B$  in  $Re$  is  $Re \cap A$ . Now,  $Re/Re \cap A$  is a projective  $R/A$  module and  $N = Re \cap A \oplus \text{Im}(M)$ . It follows that  $\text{Im}(M) = (Re/Re \cap A) \oplus Y$  for some  $R/A$ -module  $Y$  and therefore  $Re/Re \cap A$  is a minimal faithful module for  $R/A$ .

### Noetherian FPF rings

In this section we prove that all Noetherian FPF rings are products of bounded Dedekind prime rings and Quasi-Frobenius rings. We use this result to correct an error on the last page of "FPF Ring Theory" concerning which group rings are FPF.

**Lemma 11.** *Let  $R$  be a Noetherian ring and  $L$  a left ideal maximal with respect to  ${}^{\perp}L \neq 0$ . Then  $L$  is a prime two sided ideal.*

*Proof.* Clearly  ${}^{\perp}L = {}^{\perp}(LR)$  so  $L = LR$ . Also, if  $A$  and  $B$  are ideals with  $A \supset L$  and  $B \supset L$  such that  $AB \subset L$ , then  ${}^{\perp}L(AB) = 0$ . So either  ${}^{\perp}B \neq 0$  or  ${}^{\perp}LA = 0$  i.e.  ${}^{\perp}A \neq 0$  which is not possible by the choice of  $L$ .

**Proposition 12.** *Let  $R$  be a ring. Then ring  $R$  is said to be right*



*essentially bounded if every right ideal which is essential contains a two sided ideal which is right essential.*

**Proposition 13.** *Let  $R$  be a right FPF ring with right singular ideal small in  $R$ . Then  $R$  is right essentially bounded.*

*Proof.* Let  $A$  be an essential right ideal and  $B$  the ideal of  $R$  contained in  $A$  which is as large as possible. Suppose  $B$  is not right essential and choose  $H$  contained in  $A$  so that  $B \cap H = 0$  and  $B \oplus H$  is right essential. Now  $E(R) = E(B) \oplus E(H)$ , so  $1 = \hat{b} + \hat{h}$  for some  $\hat{b} \in E(B)$  and  $\hat{h} \in E(H)$ . Next consider  $\hat{b}R + \hat{h}R = M \supset R$ . We have  $\hat{b}B = B$  and  $\hat{h}B = 0$ . Moreover,  $M/H \supset R/H$  which is faithful since  $H$  contains no non-zero ideals. But the trace in  $R$  of  $M/H$  is the trace of  $\hat{b}R$  plus the trace of  $\hat{h}R/H$ , and the trace of  $\hat{h}R/H$  is in the right singular ideal since  $H$  is essential in  $E(H)$ . We have supposed the right singular ideal is small, so we have  $R = \text{trace}(\hat{b}R)$ . Now any map  $f: \hat{b}R \rightarrow \hat{h}R$  has  $B$  in the kernel since  $B$  is two sided. This implies  $\hat{h}R$  is singular and in particular that  $h$  is singular. Finally, take  $E$  an essential right ideal which contains  $B$  and  $\hat{h}E = 0$ . Then  $E = 1E = \hat{b}E$ , so  $E \subset E(B)$  and  $B$  is right essential in  $E$  which is right essential in  $R$ , giving us that  $B$  was right essential.

The next result is generalization of a result of T.G. Faticoni [Fa].

**Proposition 14.** *Let  $R$  be a ring with small right singular ideal and  $P$  a right annihilator two sided ideal. If  ${}^+P$  is finitely generated as a left ideal and  $R$  is right essentially bounded then  $R/P$  is right bounded.*

*Proof.* Let  $A = {}^+P$ . Since  $A$  is finitely generated by  $a_1, \dots, a_l$ , say, as a left ideal we have that  $R/P$  embeds in  $A^l$  as a right  $R$  module. Let  $H$  be an essential right ideal of  $R/P$  and  $H_1$  the image of  $H$  in  $A^l$  under the imbedding. Let  $I$  be the image of  $R/P$  in  $A^l$  and  $M$  a submodule of  $A^l$  such that  $H \oplus M$  is essential in  $A^l$ . The intersection of  $H_1 + M$  with each  $A$  in the direct sum  $A^l$  is nonzero and essential in  $A$ .  $A$  is a two sided ideal so each intersection contains a two sided ideal which is essential. We have that  $A^l \supset V$  where  $V$  is an  $R$  bimodule and is an essential submodule of  $A^l$  and  $H_1 \oplus M$ . Now there is a right essential right ideal  $H_2$  of  $R$  such that  $A^l H_2 \subset V$  and  $H_2 \cong P$  since  $AP = 0$  and  $V$  is essential and nonzero. It follows that  $(R/H)^\perp \neq 0$  in  $R/P$  so  $H$  contains a nonzero two sided nonzero ideal of  $R/P$ .

**Lemma 15.** *Let  $R$  be a left FPF left Noetherian ring. If  $L_1, \dots, L_n$  is a set of maximal right annihilator ideals with  $L_i \neq L_j$ , for  $i \neq j$ , then  $R = L_i + \bigcap_{i \neq j}^n L_j$  and the sum  $\sum_{i=1}^n {}^{\perp}L_i$  is direct.*

*Proof.* For any  $i$  we have  $L_i \subset \bigcap_{i \neq j}^n L_j$  so that  $L_i + \bigcap_{i \neq j}^n L_j$  is a generator. But since each  $L_i$  is an annihilator each  $L_i$  is invariant so that the trace of  $L_i + \bigcap_{i \neq j}^n L_j$  is itself, that is,  $R = L_i + \bigcap_{i \neq j}^n L_j$ . The fact that  $\sum_{i=1}^n {}^{\perp}L_i$  is direct follows easily.

Next we need a result from the past. Namely, that every FPF ring splits into a product of a semiprime ring and a ring with essential left(right) singular ideal, see [P<sub>1</sub>]. For Noetherian rings their singular ideals are nilpotent. This gives us the following reduction.

**Theorem 16.** *A Noetherian FPF ring  $R$  is a product  $R_1 \times R_2$  where  $R_1$  is a finite product of bounded Dedekind primes rings and  $R_2$  is a Noetherian FPF ring with essential (on both sides) prime radical.*

We now assume that the prime radical is essential on both sides. This means that for any prime  $P$  of  $R$  that  $P$  is essential and  ${}^{\perp}P$  is contained in the right singular ideal hence in the prime radical.

Let  $\Gamma = \{P: P \text{ is a maximal right annihilator ideal}\}$ . Let  $K = \bigcap \{P: P \in \Gamma\}$ . Then  $K$  contains the prime radical and each  ${}^{\perp}P$  is contained in the prime radical so that  $\sum_{P \in \Gamma} {}^{\perp}P \subset K$ .

**Theorem 17.** *Let  $R$  be a Noetherian FPF ring with essential prime radical  $B(R)$ . Then  $B(R) = K = Z_l(R) = Z_r(R)$  and  $R/B(R)$  is a finite product of fully bounded Dedekind prime rings. Moreover,  $R$  is fully bounded.*

*Proof.* Let  $W = \sum_{P \in \Gamma} {}^{\perp}P$ . Then  $W^{\perp} \subset K$ . Now if  $W$  is essential, then certainly  $W^{\perp} \subset Z_l(R)$ . If not we must consider left ideals  $L \subset Z_l(R)$  for which  $W \cap L \neq 0$ . But as we have observed above  $W$  is contained in  $B(R)$ . We first show there are no nontrivial two sided ideals  $V$  such that  $V \cap W = 0$  and  $0 \neq V \cap K$ . Assume some such  $V$  exists. We can assume  $V \subset Z_l(R)$  because  $Z_l(R)$  is essential. Since  $R$  is right Noetherian,  $V$  is nilpotent and so  $V^{\perp} \neq 0$ . Pick such a  $V$  so that  $V^{\perp} = B$  is maximal. Now  $B \subset P$  for some  $P \in \Gamma$ . So  ${}^{\perp}B \supset {}^{\perp}P$ . Now  ${}^{\perp}P \subset W$  so  $V \cap {}^{\perp}P = 0$ . Then  $P/{}^{\perp}P \oplus R/V$  is a generator (on either side actually). The trace of  ${}_R(R/{}^{\perp}P) = ({}^{\perp}P)^{\perp}$  and the trace of  ${}_R(R/V) = B$ . Hence  $B + ({}^{\perp}P)^{\perp} = B + P = R$ , a contradiction. We now have that if  $W$  is not essential and  $L$  is maximal with respect to  $L \cap W = 0$ , then  $L$  contains

no two sided ideals, since  $Z_i(R)$  is left essential. It follows that  $R/L$  is faithful and a generator. Now under the canonical map  $R \rightarrow R/L$ ,  $W$  embeds in  $R/L$  as an essential submodule. Let  $k \in K$ . Then any map of  $R/L$  to  $Rk$  must contain  $W$  in its kernel. This means  $K$  is contained in the left singular ideal. Therefore  $K$  is the left singular ideal. The fact that  $K \supset B(R)$  gives us that  $K = B(R)$ . That  $K$  is the right singular submodule follows by symmetry. That  $R/B(R)$  is fully bounded follows since using Lemma 15 and Proposition 14 and their symmetrical statements  $R/B(R)$  is a product of prime rings which are both left and right bounded such that each left (right) ideal is a generator, hence each is FPF by 4.7 and 4.10 of [F & P].

**Theorem 18.** *If  $R$  is a Noetherian FPF ring  $R$  is a finite product of bounded Dedekind prime rings and Quasi-Frobenius rings.*

*Proof.* By using Theorem 18 and a modified version of the proof of Theorem 5.7 of [F & P] (just take the  $A$  there to be  $N =$  the nilradical) we can use Robson's decomposition theorem [Ro] and we are done.

We can apply the last theorem to group rings over Noetherian FPF rings. The example on the last page of the text of F & P is not valid. In fact almost the opposite seems to be true, at least for Noetherian rings.

**Theorem 19.** *Let  $R$  be a Noetherian FPF ring and  $G$  a finite group. If the order of  $G$  is a unit in  $R$ , then the group ring  $RG$  is FPF.*

*Proof.* By Theorem 7 we can write  $R$  as  $R_1 \times R_2$  where  $R_2$  is Quasi-Frobenius. Now  $R_2G$  is FPF by Theorem 5.26 of [F & P]. By W. Dicks [D]  $R_2G$  is hereditary and semiprime by I. Connell [Co]. By Chatters [Ch]  $RG$  is a product of prime Noetherian rings and by the results of Lorenz and Passman [L & P] since  $R$  is bounded, so is  $RG$ , hence FPF.

Note that if the order of  $G$  is only regular in  $R$  and  $R$  is prime FPF Dick's results show  $RG$  is not hereditary but it still is semiprime noetherian so can't be FPF.

**Remark.** During the preparation of this paper the author received the preprint of Faticoni's [Fa] in which Faticoni proves Theorem 6. The two proofs are quite different.

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