Math. J. Okayama Univ. 30 (1988), 71–78

CHAIN CONDITIONS AND QUOTIENT RINGS OF P.P. RINGS

Dedicated to Professor Tosiro Tsuzuku on his 60th birthday

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The purpose of the present paper is to generalize several results of R. Yue Chi Ming [9]. First, we deal with right hereditary, right Noetherian rings with some conditions (Propositions 1, 2). Second, we present a characterization of duo rings all of whose factor rings are QF (Proposition 3). Besides, a characterization of right Artinian rings by means of nonzero factor rings is given (Proposition 4). Right self-injective, strongly regular rings with nonzero socles are also considered (Proposition 5). Finally, we give a sufficient condition for the classical right quotient ring of a right p.p. ring to be von Neumann regular (Theorem 1).

Throughout this paper we shall assume, unless otherwise specified, all rings under consideration have an identity element. Subrings of a ring R are required to contain the same identity element as R and an ideal of R means a two-sided ideal. The letter R always stands for a ring and all (right or left) R-modules are supposed to be unital. The prime radical of R is denoted by $P(R)$. We call R a *normal ring* if every idempotent of R is central. A ring R is a *right* (resp. *left*) *p.p. ring* if every principal right (resp. left) ideal of R is projective; R is a *p.p. ring* if R is both right and left p.p. For any (right or left) R-module M, $Z(M)$ and $\mathrm{Soc}(M)$ denote the singular submodule and the socle of M respectively. When the right socle $\mathrm{Soc}(R_R)$ and the left socle $\mathrm{Soc}({}_RR)$ of R coincide, we write $\mathrm{Soc}(R)$ for them. A right R-module M is said to be *essentially finitely generated* if M has a finitely generated submodule which is essential in M. For any non-empty subset X of R, $r_R(X)$ (resp. $l_R(X)$) denotes the set of right (resp. left) annihilators of X in R. A right (resp. left) ideal of the form $r_R(X)$ (resp. $l_R(X)$) for some non-empty subset X of R, is called a *right* (resp. *left*) *annihilator*. A ring R is a *right Utumi ring* if R is right non-singular and every closed right ideal of R is a right annihilator. A ring R is a *right Ore ring* if R has a classical right quotient ring. We denote the maximal right quotient ring of R by $Q^r_{\max}(R)$ (or $Q_{\max}$).

In the first place we consider the maximal right quotient ring of a right Ore, normal p.p. ring.

Lemma 1. *Let R be a right Ore, normal p.p. ring. Then $Q_{\max}^{r}(R)$ is a strongly regular ring. Consequently, R is a right Utumi ring.*

Proof. Let Q be a classical right quotient ring of R. By [3, Theorem 1], Q is a strongly regular ring. Hence $Q_{\max}^{r}(Q) = Q_{\max}^{r}(R)$ is a strongly regular ring [7, XII, Corollary 5.3, p. 254]. The last part follows from [7, XII, Proposition 4.7, p. 251].

The above lemma provides us a slight generalization of [9, Corollary 2.2].

Proposition 1. *Let R be a normal, right Ore, right hereditary ring. Then the following are equivalent:*

1) *R is right Noetherian.*
2) *R satisfies ACC on right annihilators.*
3) *Every essential right ideal of R is essentially finitely generated.*

Proof. 1) ⇒ 2). Trivial.

2) ⇒ 3). First note that R is a right Utumi ring (Lemma 1). Then, by hypothesis, R satisfies ACC on closed right ideals, or equivalently, R_R is finite-dimensional [2, Theorem 3.14, p. 81]. It is now easy to see that every essential right ideal of R is essentially finitely generated.

3) ⇒ 1). This follows from [9, Proposition 2].

If every right (resp. left) ideal of R is an ideal, then R is called a *right* (resp. *left*) *duo ring*. R is a *duo ring* if R is a left duo ring as well as a right duo ring. As is well known, a right duo ring is a normal, right Ore ring.

Lemma 2. *Let R be a right duo, right non-singular ring. If R is a subring of a ring Q and R_R is essential in Q_R, then Q is normal. In particular, if Q is a maximal right quotient ring of R, then Q is strongly regular.*

Proof. For any $q \in Q$, let $(R : q) = \{r \in R \mid qr \in R\}$, which is an essential right ideal of R. Since R is a right duo ring, we have $Rqa \subseteq qaR \subseteq qR$ for every $a \in (R : q)$. Let e be an arbitrary idempotent of Q and let $a \in (R : e)$. From what we have seen above we get, for every $r \in R$, $rea \in eR$ and so $(1-e)rea = 0$. Since $Z(Q_R) = 0$, it follows that $(1-e)re = 0$. Similarly we have $er(1-e) = 0$; hence $er = re$. Thus e is commutative with every element of R, and hence is contained in the center of Q. Therefore Q is normal. The last part is obvious.

Corollary 1. *Let R be a duo, right non-singular ring. Then the maximal right and the maximal left quotient rings of R coincide.*

Proof. Obvious from the above lemma and [7, XII, Proposition 5.5, p. 255].

A right R-module M is said to be *p-injective* if, for any principal right ideal A of R, every R-homomorphism of A to M can be extended to an R-homomorphism of R to M. For divisible modules and torsion-free modules, the reader is referred to [6].

By the aid of Lemma 2 we can prove the following proposition which contains [9, Remark 3].

Proposition 2. *Let R be a duo ring. Then the following are equivalent:*

1) *R is right hereditary and (right) Noetherian.*

2) *R is right p.p. and every p-injective right R-module is injective.*

3) *Every divisible singular right R-module and every divisible torsion-free quasi-injective right R-module are injective.*

Proof. Assume 1). By [7, XV, Proposition 4.7, p. 289], R has a classical right quotient ring Q which is right Artinian. By hypothesis, the maximal right quotient ring $Q_{\max}$ of R is a strongly regular ring (Lemma 2). Being a subring of $Q_{\max}$, Q is reduced and hence is a finite direct sum of division rings. Q is also a classical left quotient ring of R, because R is a duo ring. Since every p-injective right R-module is divisible, the implications 1) $\Rightarrow$ 2) and 1) $\Rightarrow$ 3) now follow from [6, Theorem 3.4]. 2) $\Rightarrow$ 1) is [9, Remark 1], while 3) $\Rightarrow$ 1) follows from [9, Corollary 5.1].

A ring R is said to satisfy the *regularity condition* if an element a of R is a non-zero-divisor of R whenever $a+P(R)$ is a non-zero-divisor of $R/P(R)$. R is said to satisfy the *full regularity condition* if an element a of R is a non-zero-divisor of R if and only if $a+P(R)$ is a non-zero-divisor of $R/P(R)$. We call R a *right T-Goldie ring* if R/T_k as well as R is a right Goldie ring for every positive integer k, where $T_k = P(R) \cap l_R(P(R)^k)$. We cite here the following theorem of L. W. Small [5, 4.1 Small's theorem, p. 26].

The following are equivalent:

1) R has a classical right quotient ring which is right Artinian.

2) R is a right T-Goldie ring satisfying the regularity condition.

3) R is a right T-Goldie ring satisfying the full regularity condition.

As an application of Small's theorem, we get the following

Lemma 3. *Let R be a right T-Goldie ring such that $R/P(R)$ is Artinian. Then R is right Artinian.*

Proof. It is easy to see that R satisfies the regularity condition. In fact, if a is an element of R such that $a+P(R)$ is a non-zero-divisor of $R/P(R)$, then a is invertible in R. Hence by Small's theorem R has a classical right quotient ring Q which is right Artinian. But then R satisfies the full regularity condition. Therefore every non-zero-divisor of R is invertible in R, that is, $R=Q$.

Following R. Yue Chi Ming [9], a right R-module M is said to be *YJ-injective* if, for any nonzero $a \in R$, there exists a positive integer n such that $a^n \neq 0$ and any R-homomorphism of a^nR to M can be extended to an R-homomorphism of R to M. If R_R is YJ-injective, then R is called a *right YJ-injective ring*. Left YJ-injective rings are defined similarly.

A finite set of orthogonal idempotents $e_1, \dots, e_n$ of R is said to be *complete* if $e_1+\dots+e_n=1$.

We are now ready to prove the following proposition which contains [9, Remark 5(1)].

Proposition 3. *For a duo ring R, the following are equivalent:*

1) *Every factor ring of R is QF.*

2) *Every factor ring of R is left YJ-injective, left Goldie as well as right YJ-injective, right Goldie.*

Proof. 1) ⇒ 2). Trivial.

2) ⇒ 1). It suffices to show that R is QF. By hypothesis $R/P(R)$ is a right Goldie ring and is a divisible right module over itself [9, Lemma 6]. Consequently $R/P(R)$ is Artinian. By Lemma 3 we see that R is right and left Artinian. Since $\bar{R}=R/P(R)$ is duo and hence is normal, we have

$$\bar{R}=\bar{e}_1\bar{R}\oplus\cdots\oplus\bar{e}_n\bar{R},$$

where $\bar{e}_1, \dots, \bar{e}_n$ is a complete set of orthogonal idempotents and $\bar{e}_i\bar{R}$ is a division ring for each i. Then $\bar{e}_1, \dots, \bar{e}_n$ can be lifted to a complete set of orthogonal idempotents $e_1, \dots, e_n$ of R. Note that $\bar{e}_1\bar{R}, \dots, \bar{e}_n\bar{R}$ is a complete representative system of isomorphism classes of simple right R-modules and $\mathrm{Hom}_R(\bar{e}_i\bar{R}, e_jR)=0$ for $i \neq j$. For every i we have

$$\begin{aligned}\mathrm{Hom}_R(\bar{e}_i\bar{R}, R) &= \mathrm{Hom}_R(\bar{e}_i\bar{R}, \oplus_{j=1}^n e_jR) \simeq \oplus_{j=1}^n \mathrm{Hom}_R(\bar{e}_i\bar{R}, e_jR)\\ &= \mathrm{Hom}_R(\bar{e}_i\bar{R}, e_iR).\end{aligned}$$

As is easily seen, any minimal right subideal of e_iR is isomorphic to $\bar{e}_i\bar{R}$. We claim that each e_iR has a unique minimal right subideal. Suppose the contrary and let A_1, A_2 be distinct minimal right subideals of e_iR. Let $\phi: A_1 \to A_2$ be an R-isomorphism. Since R is right YJ-injective and A_1 is minimal, ϕ can be extended to an endomorphism of R_R. Hence there exists a $b \in R$ such that for any $a \in A_1$, $\phi(a) = ba$, which belongs to A_1 by hypothesis. We then have $A_1 = A_2$, contradiction. Therefore

$$\mathrm{Hom}_R(\bar{e}_i\bar{R}, e_iR) \simeq \mathrm{Hom}_R(\bar{e}_i\bar{R}, \bar{e}_i\bar{R}) \simeq \bar{e}_i\bar{R},$$

which is a simple left R-module. Similarly the R-dual of any simple left R-module is a simple right R-module. Consequently by [1, 24.4 Theorem and Definition (c), p. 204], R is QF.

If R is a semiprime ring, then the right socle and the left socle of R coincide. Moreover, the homogeneous components of the socles coincide and these are simple rings [4, IV, § 3, Theorem 1, p. 65]. By the aid of these remarks and Lemma 3, we are able to prove that [9, Remark 5(2)] holds for non-commutative rings.

Proposition 4. *A ring R is right Artinian if and only if every nonzero factor ring of R is a right Goldie ring with nonzero right socle.*

Proof. It suffices to prove the if part. By hypothesis $\bar{R} = R/P(R)$ is a right Goldie ring with nonzero (right) socle. The socle of $\bar{R}$ has finitely many homogeneous components $\bar{R}_1, \ldots, \bar{R}_n$ and these are simple Artinian rings. Let $\bar{e}_i$ be the identity element of $\bar{R}_i$ for each i. Then $\bar{e}_1, \ldots, \bar{e}_n$ are orthogonal central idempotents of $\bar{R}$. Put $\bar{e} = \bar{e}_1 + \cdots + \bar{e}_n$. If $\bar{e} \neq \bar{1}$ $(= 1 + P(R))$, then by hypothesis the right socle of $(\bar{1} - \bar{e})\bar{R}$ is nonzero, which yields a contradiction. Therefore $\bar{e} = \bar{1}$, and hence $\bar{R}$ is Artinian. By Lemma 3 we conclude that R is right Artinian.

If R is a commutative ring which contains an injective maximal ideal, then R is self-injective [9, Remark 10(4)]. As an extension of this to normal rings, we have the following

Lemma 4. *Let R be a normal ring.*

(1) *If U is a non-nilpotent minimal right ideal of R, then U_R is injective.*

(2) *If R contains an injective maximal right ideal, then R is right self-injective.*

Proof. (1) By hypothesis there exists an idempotent e of R such that $U = eR$. Since R is normal, U is an ideal of R. Let A be a nonzero right ideal of R and let $\phi: A \to U$ be a nonzero R-homomorphism. Since ϕ is surjective and U_R is projective, there exists a right ideal B of R such that $A = B \oplus \mathrm{Ker}\ \phi$. Let b be an element of B such that $\phi(b) = e$. Then $B = bU \subseteq U$, and so $B = U$. Evidently $e \cdot \mathrm{Ker}\ \phi = 0$, that is, $\mathrm{Ker}\ \phi \subseteq (1-e)R$. Since ϕ induces an automorphism of U_R, there exists an $x \in U$ such that $\phi(u) = xu$ for every $u \in U$. We then have $\phi(a) = xa$ for every $a \in A$, showing that U_R is injective.

(2) Let I be an injective maximal right ideal of R and let U be a right ideal of R such that $R = I \oplus U$. Then U is a non-nilpotent minimal right ideal. By (1) we see that U_R is injective. This completes the proof.

A right ideal A of R is said to be *von Neumann regular* in R if for every $a \in A$, there exists a $b \in R$ such that $aba = a$.

The following proposition contains [9, Remark 12].

Proposition 5. *For a normal ring R, the following are equivalent*:

1) *R is a right self-injective, von Neumann regular ring with non-zero socle.*

2) *R contains an injective maximal right ideal which is von Neumann regular in R.*

3) *R contains a non-singular, injective maximal right ideal.*

Proof. 1) ⇒ 2). Trivial.

2) ⇒ 3). Let I be an injective maximal right ideal which is von Neumann regular in R. For any $a \in Z(I_R)$, there exists a $b \in R$ such that $aba = a$. Then we have $r_R(a) = r_R(ba) = (1-ba)R$, which is an essential right ideal as well as a direct summand of R_R. Hence $r_R(a) = R$, that is, $a = 0$. Thus $Z(I_R) = 0$.

3) ⇒ 1). Since R is right self-injective (Lemma 4(2)), R is von Neumann regular [9, Theorem 13]. It is almost obvious that the socle of R is nonzero.

The following lemma seems to be well known, so we omit the proof.

Lemma 5. *Let R be a semiprime ring and let A be an ideal of R. If A is essential in R as an ideal, then A is essential in R both as a right ideal and as a left ideal.*

Lemma 6. *Let R be a right non-singular ring and let A be an ideal of R. If A is essential as a right ideal, then the centralizer $C_R(A)$ of A in R is the center of R.*

Proof. Suppose that $C_R(A)$ is not the center of R and let x be an element of $C_R(A)$ such that $xr \neq rx$ for some $r \in R$. By hypothesis there exists an $a \in A$ with $(xr-rx)a \neq 0$. But in view of $ra \in A$ we have $(xr-rx)a = x(ra)-(ra)x = 0$, contradiction. This completes the proof.

Following R. Yue Chi Ming [8], a ring R is called a *right CM-ring* if for any maximal essential right ideal A of R and any complement right subideal B of A, we have $AB \subseteq B$.

Let R be a right p.p. ring with classical right quotient ring Q. Then Q is a right p.p. ring [3, Proposition 1]. If moreover Q is right p-injective, then Q is von Neumann regular [9, Theorem 14]. We now consider the case where Q is a right CM-ring.

Theorem 1. *Let R be a right p.p. ring with classical right quotient ring Q. If Q is a right CM-ring, then Q is von Neumann regular.*

Proof. First note that Q is a semiprime ring [8, Lemma 1.6 (3)].

Case 1. $\mathrm{Soc}(Q)$ is a direct summand of ${}_QQ_Q$. We write $Q = \mathrm{Soc}(Q) \oplus I$, where I is an ideal of Q. In this case $\mathrm{Soc}(Q)$ is a semisimple Artinian ring and I is a semiprime ring. By [8, Lemma 1.6 (4), (5)] we see that I is a right non-singular, right CM-ring and that either the socle of I is nonzero or I is reduced. Since $\mathrm{Soc}(I) = \mathrm{Soc}(I_Q) = 0$, I must be reduced. Now I is a normal p.p. ring in which every non-zero-divisor is invertible, and so I is a strongly regular ring [3, Theorem 1]. Therefore Q is a von Neumann regular ring.

Case 2. $\mathrm{Soc}(Q)$ is not a direct summand of ${}_QQ_Q$. Let I be an ideal of Q such that $K = \mathrm{Soc}(Q) + I = \mathrm{Soc}(Q) \oplus I$ is essential in ${}_QQ_Q$. By Lemma 5 we see that K is essential in Q_Q. If K contains a nonzero nilpotent element, then K is a direct summand of Q_Q [8, Lemma 1.6 (1)], and hence $K = Q$, contradiction. Consequently K is reduced. We now wish to show that Q is normal. For this, let e be an idempotent of Q and let a be an arbitrary element of K. Then $ea - eae \in K$ and $(ea-eae)^2 = 0$, whence $ea = eae$. Similarly we have $ae = eae$. Thus e is commutative with every element of K and hence is contained in the center of Q (Lemma 6). Therefore Q is normal. In view of the fact that Q is a normal p.p. ring in which every non-zero-divisor is invertible, we see that Q is a strongly regular ring [3, Theorem1]. This completes the proof.

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(Received May 30, 1987)