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CORRESPONDENCES OF MODULES OVER A MORITA RING

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The Theorem of Morita on correspondences of modules over unital rings was generalized in case of non-unital rings in [1] and [2]. It deals with modules M over a ring R which satisfy $RM = M$. Modules with this property are said to be lower closed. The concept of lower and upper closed ideals are introduced in [2]. Let J be a left ideal of R. We say that J is lower closed if $RJ = J$ and that J is upper closed if $R^{-1}J = J$ where $R^{-1}J = \{r \in R \mid Rr \subseteq J\}$. In this paper, we introduce the concept of upper closed modules and generalize the theorem of correspondences in this case. A definition of upper closed modules is given in 1. Let M be a (left) R-module. When $Rm = 0$ for $m \in M$ implies $m = 0$, we say that M is a proper R-module. Generally, let $m \in M$. Then, m induces an R-homomorphism of R to M by its right multiplication which we denote by $\overline{m}$: $r^{\overline{m}} = rm$. Let $\overline{M} = \{\overline{m} \mid m \in M\}$. Then, M is homomorphic to $\overline{M}$. That M is proper is equivalent to that this homomorphism is an isomorphism. When M is proper and $\overline{M}$ coincides with $\mathrm{Hom}_R(R, M)$, we say that M is upper closed. In 2, we introduce a Morita ring and modules over a Morita ring. A Morita ring is just a Morita context considered as a ring. Sometimes, it is called a ring of a Morita context. A ring C is a Morita ring if $C = C_{11} \oplus C_{22} \oplus C_{12} \oplus C_{21}$ (a direct sum of submodules C_{ij}), where $C_{ij}C_{jk} \subseteq C_{ik}$ and $C_{ij}C_{mn} = 0$ if $j \neq m$. Thus, C_{ii} are rings and C_{ij} are C_{ii}-C_{jj}-bimodules. Now, let W be a C-module. We say that W is a module over a Morita ring C, or shortly, a C_m-module in case that $W = M_1 \oplus M_2$ (a direct sum of submodules M_i) where $C_{ij}M_j \subseteq M_i$ and $C_{ij}M_k = 0$ if $j \neq k$. Then, M_1 and M_2 are called the first and second components of W. Naturally, M_i are C_{ii}-modules. In this case, we say that M_1 and M_2 correspond to each other via a C_m-module W. This is a general concept of correspondences of modules. The correspondence is however not one to one. In 3 and 4, we restrict our modules to closed modules and show that the correspondence is one to one. We also show that if (closed modules) M and N correspond to each other, M' and N' correspond to each other, and there is a homomorphism ϕ_1 of M to M', then there is a uniquely determined homomorphism ϕ_2 of N to N'. If M'' and N'' correspond to each other and if there is a homomorphism ϕ_1' of M' to M'' with the homomorphism ϕ_2' of N' to N'' determined by ϕ_1', then $\phi_2 \circ \phi_2'$ is the homomorphism of N to N'' determined by the homomorphism $\phi_1 \circ \phi_1'$ of M to M''.

1. Upper closed modules. Let M be an R-module. In the following, we denote $M^* = \mathrm{Hom}_R(R, M)$. M^* is an R-module. Here, rf for $r \in R$ and $f \in M^*$ is defined to be a mapping of R to M such that $x^{rf} = (xr)^f$. Let $m \in M$. The mapping $x \rightarrow xm$ is an R-homomorphism of R to M, which we denote by $\bar{m}$. So, $\bar{m} \in M^*$. Especially, $rf = \overline{r^f}$ since $(xr)^f = x(r^f)$ in the above. Let $\bar{M} = \{\bar{m} \mid m \in M\}$. $\bar{M}$ is an R-submodule of M^*. The mapping $m \rightarrow \bar{m}$ is an R-homomorphism of M to $\bar{M}$. When this is an isomorphism, i.e., one to one, we say that M is a proper R-mdoule. In other words, M is proper if and only if $Rm = 0$ implies $m = 0$ for $m \in M$.

Proposition 1. *If M is proper, so is M^*.*

Proof. Let f be an element of M^* such that $Rf = 0$. Then, $0 = rf = \overline{r^f}$ for any $r \in R$. Since M is proper, $r^f = 0$, and hence $f = 0$. M^* is proper.

Proposition 2. *Suppose that M is proper. There exists a proper R-module U such that* (1) *$RU \subseteq M \subseteq U$ and* (2) *if E is a proper R-module with $RE \subseteq M \subseteq U \subseteq E$, then $E = U$. Moreover, if V is another proper R-module satisfying* (1) *and* (2), *then U and V are isomorphic over M.*

Proof. M is isomorphic with $\bar{M}$. We prove Proposition 2 for $\bar{M}$. We show that M^* satisfies (1) and (2) (for $\bar{M}$ in place of M). M^* is proper by Proposition 1 and naturally contains $\bar{M}$. If $r \in R$ and $f \in M^*$, then $rf = \bar{m}$ with $m = r^f$ as we noted above. So, (1) holds. For (2), let E be a proper R-module with $RE \subseteq \bar{M} \subseteq M^* \subseteq E$. Let $g \in E$. Then, $rg \in \bar{M}$ for any $r \in R$, and hence $rg = \overline{m(r)}$ with $m(r) \in M$. The mapping $r \rightarrow m(r)$ is an R-homomorphism of R to M, and hence there is an element f of M^* such that $m(r) = r^f$. $\overline{m(r)} = \overline{r^f} = rf$ as above. Thus, $rg = \overline{m(r)} = rf$. $r(g-f) = 0$. Here, $g-f$ is an element of E, which is proper. So, $g-f = 0$, or $g = f$. This implies $E \subseteq M^*$. Therefore, $E = M^*$, and (2) holds. Lastly, suppose that V is another proper R-module satisfying (1) and (2) for $\bar{M}$. As above, we can conclude that if $v \in V$, then $rv = rf(v)$ with some $f(v) \in M^*$. The mapping ϕ: $v \rightarrow f(v)$ is an R-isomorphism of V into M^* over $\bar{M}$. $\phi(V)$ satisfies (1) and (2) as V does. Now, observe that M^* satisfies the condition of E in (2) for $\phi(V)$. Thus, $\phi(V) = M^*$, which implies that V and M^* are isomorphic over $\bar{M}$.

We call U in Proposition 2 an upper closure of a proper R-module M. We denote an upper closure of M by $R^{-1}M$. $R^{-1}M$ is determined up to isomorphisms over M.

Definition 1. We say that M is upper closed if M is proper and $M = R^{-1}M$.

When $R^2 = R$, $R^{-1}M$ is upper closed for any proper R-module. For, let $N = R^{-1}(R^{-1}M)$. N is proper as well as $R^{-1}M$. $RN = R(RN) \subseteq R(R^{-1}M) \subseteq M \subseteq R^{-1}M \subseteq N$. Thus, by (2) of Proposition 2, $N = R^{-1}M$, which shows that $R^{-1}M$ is upper closed.

2. Morita rings; Modules over a Morita ring. A Morita ring is a ring C such that $C = C_{11} \oplus C_{22} \oplus C_{12} \oplus C_{21}$ where C_{ij} are submodules satisfying $C_{ij}C_{jk} \subseteq C_{ik}$ and $C_{ij}C_{mn} = 0$ if $j \neq m$. Then, C_{ii} are subrings of C and C_{ij} are C_{ii}-C_{jj}-bimodules. For example, the matrix ring $M_2(R)$ of 2×2 matrices over a ring R is a Morita ring where $C_{ij} = Re_{ij}$ (e_{ij} are the matrix units). Usually, the system $\langle C_{11}, C_{22}, C_{12}, C_{21} \rangle$ is called a Morita context. In this paper, we consider a Morita ring C which satisfies the additional properties $C_{12}C_{21} = C_{11}$ and $C_{21}C_{12} = C_{22}$. For example, if J is a left ideal of R, $RJe_{11} \oplus JRe_{22} \oplus Re_{12} \oplus Je_{21}$, which is a subring of $M_2(R)$, is a Morita ring satisfying the above properties. Note also that if C satisfies the properties then we have $C_{11}C_{12} = C_{12}C_{22}$ and $C_{22}C_{21} = C_{21}C_{11}$ as we can easily verify.

Definition 2. Let W be a C-module. We say that W is a module over a "Morita" ring, or shortly, a C_m-module if there exists submodules M_1 and M_2 of W such that $W = M_1 \oplus M_2$, $C_{ij}M_j \subseteq M_i$ and $C_{ij}M_k = 0$ if $j \neq k$.

In the above definition, we should have called W as a C_m-module with the components M_1 and M_2. When this is the case, we say that a C_{11}-module M_1 and a C_{22}-module M_2 correspond to each other via a C_m-module W. The correspondence is generally not one to one. First, we want to show the existence of the correspondence for a given M_1. Let M be a C_{11}-module. We need to show the existence of a C_m-module $W = M_1 \oplus M_2$ such that $M_1 = M$. For the purpose, consider $\mathrm{Hom}_{C_{11}}(C_{12}, M)$, which we denote by $M*$ throughout this paper. $M*$ is a C_{22}-module. Here, we define a product bf for $b \in C_{22}$ and $f \in M*$ as a mapping $u \to (ub)^f$, which is seen to be a C_{11}-homomorphism of C_{12} to M. Now, consider $M \oplus M*$, a direct sum of modules. By defining left multiplication by elements of C as follows, we can obtain a C_m-module $W = M \oplus M*$. In the following, denote elements of C_{11} by a, elements of C_{22} by b, elements of C_{12} by u, elements of C_{21} by v, elements of M by m and elements of $M*$ by f. Define the products am and bf in a natural way (via a C_{11}-module M and a C_{22}-module $M*$). We define $af = bm = um = vf = 0$. Let vm be defined as an element of $M*$ such that $u^{vm} = (uv)m$. Lastly, define $uf = u^f$. Now, using the linearity, we can define cw for any $c \in C$ and $w \in W$. It is routine to verify that W becomes a C_m-module. In the following, we denote $W = M \oplus M*$ constructed above by W_M.

Definition 3. A C_m-module $W = M_1 \oplus M_2$ is said to be proper if M_i are proper C_{ii}-modules ($i = 1, 2$).

Proposition 3. *If M is a proper C_{11}-module, then W_M is proper.*

Proof. We need only to show that $M*$ is a proper C_{22}-module. Let $f \in M*$ and suppose that $C_{22}f = 0$. $bf = 0$ for any b. So, $0 = u^{bf} = (ub)^f$. Now for any $a \in C_{11}$, we have $au = \sum (u_t b_t)$ with some $u_t \in C_{12}$ and $b_t \in C_{22}$ because $C_{11}C_{12} = C_{12}C_{22}$. Thus, $a(u^f) = (au)^f = \sum (u_t b_t)^f = 0$. Since M is proper, this implies that $u^f = 0$, and hence $f = 0$. We have shown that $M*$ is proper.

3. Lower closed proper C_m-modules.

Definition 4. A C_m-module $W = M_1 \oplus M_2$ is said to be lower closed if $C_{ij}M_j = M_i$ for all i and j.

Note that if W is lower closed then M_i are lower closed but the converse is not true.

Theorem 1. *Let M be a lower closed proper C_{11}-module. There exists a lower closed proper C_m-module $W = M_1 \oplus M_2$ such that $M_1 = M$.*

Proof. Consider $W_M = M \oplus M*$, which is a proper C_m-module by Proposition 3. Let $N = C_{21}M$. N is clearly a C_{22}-submodule of $M*$. We have $C_{22}N = C_{22}C_{21}M = C_{21}C_{11}M = C_{21}M = N$, $C_{12}N = C_{12}C_{21}M = C_{11}M = M$ and $C_{21}M = C_{21}C_{12}N = C_{22}N = N$. Let $W = M \oplus N$. It is clear that W is a lower closed proper C_m-module.

Theorem 1 implies that for a given lower closed proper C_{11}-module, there exists a lower closed proper C_{22}-module such that the two modules correspond to each other via a lower closed proper C_m-module. Next, we want to show that this correspondence is one to one up to within isomorphisms (of modules).

Proposition 4. *Let $W = M \oplus N$ and $W' = M' \oplus N'$ be both proper C_m-modules. Suppose that $N = C_{21}M$. If ϕ_1 is a C_{11}-homomorphism of M to M', then there exists uniquely a C_{22}-homomorphism ϕ_2 of N to N' such that the mapping $\phi = \phi_1 \oplus \phi_2$ of W to W' obtained from ϕ_1 and ϕ_2 in a natural way is a C-homomorphism.*

Proof. Let $n \in N$. Since $N = C_{21}M$, $n = \sum v_t m_t$ with $v_t \in C_{21}$ and $m_t \in M$. Define a mapping ϕ_2 of N to N' by $n^{\varphi_2} = \sum v_t(m_t^{\varphi_1})$. We have to show that ϕ_2 is well defined. For it, it is sufficient to show that $\sum v_t m_t = 0$ implies $\sum v_t(m_t^{\varphi_1}) = 0$. Assume $\sum v_t m_t = 0$. Then, for any u, $0 = u(\sum v_t m_t) = \sum (uv_t) m_t$, whence $0 = (\sum (uv_t) m_t)^{\varphi_1} = \sum (uv_t)(m_t^{\varphi_1}) = u(\sum v_t(m_t^{\varphi_1}))$. This implies that $b(\sum v_t(m_t^{\varphi_1})) = 0$ for any $b \in C_{22}$ since $C_{22} = C_{21}C_{12}$. Since N' is proper, we have $\sum v_t(m_t^{\varphi_1}) = 0$ as required. Now, let $\phi = \phi_1 \oplus \phi_2$ be the mapping of W to W' such that $\phi = \phi_1$ on M and $\phi = \phi_2$ on N. We can verify that ϕ is a C-homomorphism of W to W'. Lastly, ϕ is uniquely determined by ϕ_1, because $n^{\varphi} = (\sum v_t m_t)^{\varphi} = \sum v_t(m_t^{\varphi}) = \sum v_t(m_t^{\varphi_1})$. Naturally, ϕ_2 is uniquely determined by ϕ_1.

Theorem 2. *Let $W = M \oplus N$ and $W' = M' \oplus N'$ be lower closed proper C_m-modules. If M and M' are C_{11}-isomorphic, then N and N' are C_{22}-isomorphic.*

Proof. Let ϕ_1 be the isomorphism of M to M'. By Proposition 4, there exists a C-homomorphism ϕ of W to W' such that it maps N to N'. Similarly, consider the isomorphism ϕ_1^{-1} of M' to M. There exists a C-homomorphism ϕ' determined by ϕ_1^{-1} of W' to W. Now, we consider $\phi_1 \circ \phi_1^{-1}$ which is the identity mapping of M to M. $\phi \circ \phi'$ is the C-homomorphism of W to W determined by $\phi_1 \circ \phi_1^{-1}$. It must be the identity mapping of W. By the uniqueness part of Proposition 4, we have $\phi \circ \phi' =$ the identity. Similarly, $\phi' \circ \phi$ must be the identity mapping of W' to W'. Thus, ϕ and ϕ' are inverse each other. Hence, when we restrict ϕ on N, we obtain a C_{22}-isomorphism of N onto N'.

We denote ϕ_2 of Proposition 4 by $F(\phi_1)$ (or, more precisely, $F_{W,W'}(\phi_1)$). Then, the following Theorem 3 is almost clear.

Theorem 3. *Suppose that $W = M \oplus N$, $W' = M' \oplus N'$ and $W'' = M''$*

$\oplus N''$ are lower closed proper C_m-modules. If ϕ_1 is a C_{11}-homomorphism of M to M' and ϕ_1' is a C_{11}-homomorphism of M' to M'', then $\phi_1 \circ \phi_1'$ is a C_{11}-homomorphism of M to M'', and we have $F(\phi_1 \circ \phi_1') = F(\phi_1) \circ F(\phi_1')$.

4. Upper closed C_m-modules.

Definition 4. A C_m-module $W = M_1 \oplus M_2$ is said to be upper closed if M_i are upper closed C_{ii}-modules $(i = 1, 2)$.

Let $W = M \oplus N$ and $W' = M' \oplus N'$ be C_m-modules. When W is a C-submodule of W' and $M \subseteq M'$ and $N \subseteq N'$, we say that W is a C_m-submodule of W'.

Proposition 5. *Let $W = M \oplus N$ be a proper C_m-module. If M' is a proper C_{11}-module containing M, then W is isomorphically imbedded as a C_m-submodule in $W_{M'}$.*

Proof. Let ϕ_1 be the imbedding mapping of M to M'. We define a mapping ϕ_2 of N to $(M')*$ by $n^{\varphi_2} = \bar{n}$ for $n \in N$ where $\bar{n}$ is the right multiplication: $u^{\bar{n}} = un$. Since N is proper, ϕ_2 is a one to one mapping. We can verify that ϕ_2 is a C_{22}-homomorphism of N to $M*$ and hence to $(M')*$. Then, $\phi = \phi_1 \oplus \phi_2$ is an isomorphism of W to $W_{M'}$. It is obvious that $\phi(W)$ is a C_m-submodule of $W_{M'}$.

Similarly, we can conclude that if $W = M \oplus N$ is a proper C_m-module and if N' is a proper C_{22}-module containing N, then there exists a proper C_m-module $M' \oplus N'$, which contains a C_m-submodule isomorphic to W.

Let $W = M \oplus N$ be a C_m-module. For a subset S of M, we define $[C_{11}^{-1}S]_W = \{m \in M \mid C_{11}m \subseteq S\}$ and $[C_{12}^{-1}S]_W = \{n \in N \mid C_{12}n \subseteq S\}$. Similarly, for a subset T of N, we define $[C_{22}^{-1}T]_W = \{n \in N \mid C_{22}n \subseteq T\}$ and $[C_{21}^{-1}T]_W = \{m \in M \mid C_{21}m \subseteq T\}$. Especially, if S is an upper closed C_{11}-module and if W is upper closed, then $[C_{11}^{-1}S]_W = C_{11}^{-1}S = S$.

Proposition 6. *Let W, S and T be as above. Then, $[C_{22}^{-1}[C_{12}^{-1}S]_W]_W = [C_{12}^{-1}[C_{11}^{-1}S]_W]_W$ and $[C_{11}^{-1}[C_{21}^{-1}T]_W]_W = [C_{21}^{-1}[C_{22}^{-1}T]_W]_W$. Also, $[C_{21}^{-1}[C_{12}^{-1}S]_W]_W = [C_{11}^{-1}S]_W$ and $[C_{12}^{-1}[C_{21}^{-1}T]_W]_W = [C_{22}^{-1}T]_W$.*

Proof. Proposition 6 follows from $C_{12}C_{22} = C_{11}C_{12}$, $C_{21}C_{11} = C_{22}C_{21}$, $C_{12}C_{21} = C_{11}$ and $C_{21}C_{12} = C_{22}$.

Theorem 4. *If M is an upper closed C_{11}-module, then $M*$ is an upper closed C_{22}-module, and hence W_M is an upper closed C_m-module.*

Proof. Let $N' = C_{22}^{-1}(M*)$, an upper closure of $M*$. There exists a proper C_m-module $W' = M' \oplus N'$ which contains W_M as a C_m-submodule by the remark right after Proposition 5. First, we want to show that $[C_{12}^{-1}M]_{W'} = M*$. Clearly, $[C_{12}^{-1}M]_{W'} \supseteq M*$. Let $y \in [C_{12}^{-1}M]_{W'}$. y is an element of N' such that $C_{12}y \subseteq M$. Let $\bar{y}$ be the right multiplication by y, i. e., $u^{\bar{y}} = uy$ for $u \in C_{12}$. $\bar{y}$ is an element of $M* = \mathrm{Hom}_{C_{11}}(C_{12}, M)$. But, by the definition of $W_M, u^{\bar{y}} = u\bar{y}$. Thus, $uy = u\bar{y}$ in W_M. Therefore, $u(y-\bar{y}) = 0$. Since W' is proper, $y-\bar{y} = 0$, or $y = \bar{y} \in M*$. We showed that $[C_{12}^{-1}M]_{W'} = M*$. Now, $[C_{22}^{-1}M*]_{W'} = [C_{22}^{-1}[C_{12}^{-1}M]_{W'}]_{W'} = [C_{12}^{-1}[C_{11}^{-1}M]_{W'}]_{W'}$ by Proposition 6. Since M is upper closed, $[C_{11}^{-1}M]_{W'} = M$. Therefore, $[C_{22}^{-1}M*]_{W'} = [C_{12}^{-1}M]_{W'} = M*$. So, $N' \subseteq [C_{12}^{-1}M]_{W'} \subseteq M* \subseteq N'$, and hence $C_{22}^{-1}(M*) = N' = M*$. We have shown that $M*$ is upper closed, and the proof of Theorem 4 is completed.

Theorem 4 implies that if M is an upper closed C_{11}-module then $M*$ is an upper closed C_{22}-module and M and $M*$ correspond to each other via an upper closed C_m-module.

Proposition 7. *Let $W = M \oplus N$ and $W' = M' \oplus N'$ be upper closed C_m-modules. If W is a C_m-submodule of W', then $[C_{12}^{-1}M]_{W'} = N$ and $[C_{21}^{-1}N]_{W'} = M$.*

Proof. Let $T = [C_{12}^{-1}M]_{W'}$. Then, $[C_{21}^{-1}T]_{W'} = [C_{11}^{-1}M]_{W'}$ by Proposition 6. Since M is upper closed, $[C_{11}^{-1}M]_{W'} = M$. Hence, $[C_{21}^{-1}T]_{W'} = M$. On the other hand, $N \subseteq T$. So, $[C_{21}^{-1}N]_{W'} \subseteq [C_{21}^{-1}T]_{W'} = M$. Clearly, $M \subseteq [C_{21}^{-1}N]_{W'}$. Therefore, $[C_{21}^{-1}N]_{W'} = M$. The first identity is similarly proved.

Theorem 5. *Let $W = M \oplus N$ and $W' = M' \oplus N'$ be upper closed C_m-modules. If M and M' are C_{11}-isomorphic, then N and N' are C_{22}-isomorphic.*

Proof. It is enough to show that if $W = M \oplus N$ is upper closed then N and $M*$ are C_{22}-isomorphic. By Proposition 5, W is imbedded in W_M as a C_m-submodule. So, we may assume that W is a C_m-submodule of W_M. Then, by Proposition 7, $[C_{12}^{-1}M]_{W_M} = N$. It is clear that $[C_{12}^{-1}M]_{W_M} = M*$. Therefore, $N = M*$. We have shown that the imbedding isomorphism gives an isomorphism of N onto $M*$.

Proposition 8. *Let $W = M \oplus N$ and $W' = M' \oplus N'$ be upper closed C_m-modules. If ϕ_1 is a C_{11}-homomorphism of M to M', then there exists uniquely a C_{22}-homomorphism ϕ_2 of N to N' such that $\phi = \phi_1 \oplus \phi_2$ is a C-homomorphism of W to W'.*

Proof. For the existence of ϕ or of ϕ_2, we may assume that $W = W_M = M \oplus M*$ and $W' = W_{M'} = M' \oplus (M')*$ by virtue of Theorem 5. We define a mapping ϕ_2 of $M*$ to $(M')*$ as follows. For $f \in M*$, let f^{φ_2} be a mapping $u \to (u^f)^{\varphi_1}$. We can show that $\phi = \phi_1 \oplus \phi_2$ is a C-homomorphism of W to W'. The uniqueness of ϕ or of ϕ_2 is proved as in Proposition 4.

Now, let $\phi_2 = F(\phi_1)$ as in case of lower closed proper modules. We can obtain the same theorem as Theorem 3 in case of upper closed modules.

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