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PROJECTIVE MODULES OVER REGULAR RINGS OF BOUNDED INDEX

Dedicated to Professor Hiroyuki Tachikawa on his 60th birthday

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In this paper, we shall study directly finite projective modules over regular rings of bounded index. In [3], we have shown that a directly finite regular ring R satisfying the comparability axiom has the property that the direct sum of two directly finite projective R-modules is directly finite. We shall call this property (DF). It is natural to ask which kind of regular rings have (DF). However, we see (Example) that there exists a commutative regular ring which does not have (DF). Therefore we shall study the property (DF) for regular rings of bounded index. First, we give a criterion of the directly finiteness of projective modules over these rings (Theorem 2), and, using this criterion, we show that any direct sums of finite copies of directly finite projective modules over these rings are directly finite (Theorem 4). In main Theorem 8, we characterize the property (DF) for regular rings R of bounded index.

Throughout this paper, R is a ring with identity and R-modules are unitary right R-modules.

§ 1. **Preliminaries.** For two R-modules P and Q, we use $P \lesssim Q$ (resp. $P \lesssim\oplus\, Q$) to mean that P is isomorphic to a submodule of Q (resp. a direct summand of Q). For a submodule P of an R-module Q, $P <\oplus\, Q$ means that P is a direct summand of Q. For a cardinal number α and an R-module P, αP denotes a direct sum of α-copies of P.

Definition. An R-module P is *directly finite* provided that P is not isomorphic to a proper direct summand of itself. If P is not directly finite, then P is said to be *directly infinite*.

Definition. The *index* of a nilpotent element x in a ring R is the least positive integer n such that $x^n = 0$. (In particular, 0 is nilpotent of index 1.) The *index* of a regular ring R is the supremum of the indices of all nilpotent elements of R. If this supremum is finite, then R is said to have *bounded index*.

Note that a regular ring R is abelian (i.e., all idempotents in R are central) if and only if it has index 1.

We shall recall the following basic properties, which we need for § 2.

(1) If P is a projective module over a regular ring, then all finitely generated submodules of P are direct summands of P ([1, Theorem 1.11]).

(2) Every projective modules over regular rings have the exchange property (see [3] and [4]).

(3) If R is a regular ring of bounded index, then it is unit-regular, and so all finitely generated projective R-modules have the cancellation property ([1, Theorem 4.14 and Corollary 7.11]).

(4) Let R be a regular ring, and let n be a positive integer. Then R has index at most n if and only if R contains no direct sums of $n+1$ nonzero pairwise isomorphic right ideals ([1, Theorem 7.2]).

(5) Let R be a regular ring of bounded index, and let P be a finitely generated projective R-module. Then $End_R(P)$ has bounded index ([1, Corollary 7.13]).

§ 2. Directly finite projective modules.

Lemma 1. *Let R be a regular ring of bounded index at most n for some positive integer n, and let B, $A_1,\dots,A_k$ be projective R-modules such that each A_i is cyclic. Let*

$$\begin{aligned} & A_{11} \oplus\cdots\oplus A_{1k} \\ \simeq\ & A_{21} \oplus\cdots\oplus A_{2k} \oplus B_2 \\ \simeq\ & \cdots\cdots\cdots \\ \simeq\ & A_{s_k1} \oplus\cdots\oplus A_{s_kk} \oplus B_{s_k} \\ \lesssim\ & A_1 \oplus\cdots\oplus A_k \oplus B, \end{aligned}$$

and let

$$\begin{aligned} & B_2 \oplus\cdots\oplus B_{s_k} <\oplus\ B \text{ and} \\ & A_{1i} \oplus\cdots\oplus A_{s_ki} <\oplus\ A_i \text{ for } i = 1,\dots,k, \end{aligned}$$

where $s_1 = 1+n$ and $s_k = 1+ns_{k-1}$ for $k > 1$. Then $A_{11} \oplus\cdots\oplus A_{1k} \lesssim\oplus\ B_2 \oplus\cdots\oplus B_{s_k} <\oplus\ B$.

Proof. Set $Q = A_{11} \oplus\cdots\oplus A_{1k}$. First we consider the case $k = 1$. Let

$$Q = A_{11}$$

$$
\begin{aligned}
&\simeq A_{21} \oplus B_2 \\
&\simeq \cdots\cdots \\
&\simeq A_{s_1 1} \oplus B_{s_1} \\
&\lesssim A_1 \oplus B,
\end{aligned}
$$

and let

$$
\begin{aligned}
&A_{11} \oplus \cdots \oplus A_{s_1 1} <\oplus A_1 \text{ and} \\
&B_2 \oplus \cdots \oplus B_{s_1} <\oplus B.
\end{aligned}
$$

Then there exist direct summands A'_{j1} of A_{j1} and B'_j of B_j for $j = 3,\ldots,s_1$ such that $A'_{j-11} \simeq A'_{j1} \oplus B'_j$, where $A'_{21} = A_{21}$. Note that $\{A_{11}, A'_{21},\ldots,A'_{s_1}\}$ is an independent family of submodules of A_1 and $A'_{s_1 1} \lesssim \cdots \lesssim A'_{21} \lesssim A_{11} = Q$. Since $A_1 \lesssim R$ and R has index at most n, we have that $A'_{s_1 1} = 0$, and so that $Q = A_{11} \simeq A_{21} \oplus B_2 \simeq \cdots \simeq B'_{s_1} \oplus \cdots \oplus B_2 <\oplus B_2 \oplus \cdots \oplus B_{s_1} <\oplus B$. Next, let $k > 1$ and assume that the lemma holds for $k-1$. Let

$$
\begin{aligned}
Q &= A_{11} \oplus \cdots \oplus A_{1k} \\
&\simeq A_{21} \oplus \cdots \oplus A_{2k} \oplus B_2 \\
&\simeq \cdots\cdots \\
&\simeq A_{s_k 1} \oplus \cdots \oplus A_{s_k k} \oplus B_{s_k} \\
&\lesssim A_1 \oplus \cdots \oplus A_k \oplus B,
\end{aligned}
$$

and let

$$
\begin{aligned}
&B_2 \oplus \cdots \oplus B_{s_k} <\oplus B \text{ and} \\
&A_{1i} \oplus \cdots \oplus A_{s_k i} <\oplus A_i \text{ for } i = 1,\ldots,k.
\end{aligned}
$$

Then there exist decompositions

$$A_{ji} = A^1_{ji} \oplus \cdots \oplus A^k_{ji} \text{ and } B_j = B^1_j \oplus \cdots \oplus B^k_j$$

for $j = 2,\ldots,s_k$ and $i = 1,\ldots,k$ such that

$$A_{1i} \simeq A^i_{j1} \oplus \cdots \oplus A^i_{jk} \oplus B^i_j.$$

Note that $\{A^i_{j1},\ldots,A^i_{jk}, B^i_j\}_{i,j}$ is an independent family of submodules of $A_1 \oplus \cdots \oplus A_k \oplus B$. We devide the set $\{2,\ldots,s_k\}$ into $\{2,\ldots,n+1;\ n+2,\ldots,2n+1;\ldots;s_k-n+1,\ldots,s_k\}$. Applying the case $k = 1$ for each part, we have direct summands $C^i_{p,m}$ of $A^i_{(p-2)n+2,m} \oplus \cdots \oplus A^i_{(p-1)n+1,m}$ and D^i_p of $B^i_{(p-2)n+2} \oplus \cdots \oplus B^i_{(p-1)n+1}$ for $p = 2,\ldots,s_{k-1}+1$ and $m = 2,\ldots,k$ such that

$$
\begin{aligned}
A_{1i} &\simeq C^i_{22} \oplus \cdots \oplus C^i_{2k} \oplus D^i_2 \\
&\simeq \cdots\cdots
\end{aligned}
$$

$$\simeq C^i_{s_{k-1}+1,2} \oplus \cdots \oplus C^i_{s_{k-1}+1,k} \oplus D^i_{s_{k-1}+1}.$$

By the induction hypothesis, we have that $C^i_{22} \oplus \cdots \oplus C^i_{2k} \lesssim D^i_3 \oplus \cdots \oplus D^i_{s_{k-1}+1}$. As a result,

$$\begin{aligned} Q &= A_{11} \oplus \cdots \oplus A_{1k} \\ &\lesssim (D^1_2 \oplus \cdots \oplus D^1_{s_{k-1}+1}) \oplus \cdots \oplus (D^k_2 \oplus \cdots \oplus D^k_{s_{k-1}+1}) \\ &<\oplus\ B_2 \oplus \cdots \oplus B_{s_k} <\oplus\ B. \end{aligned}$$

Thus the induction works and the lemma is proved.

Theorem 2. *Let R be a regular ring of bounded index. For a projective R-module P with a cyclic decomposition $P = \bigoplus_{\lambda \in \Lambda} P_\lambda$, the following conditions* (a)—(d) *are equivalent*:

(a) *P is directly infinite.*

(b) *There exists a nonzero cyclic projective R-module X such that $\aleph_0 X \lesssim P$.*

(c) *There exists a nonzero cyclic projective R-module X such that $X \lesssim \bigoplus_{\lambda \in \Lambda - \{\lambda_1, \ldots, \lambda_n\}} P_\lambda$ for all finite subsets $\{\lambda_1, \ldots, \lambda_n\}$ of Λ.*

(d) *There exists a nonzero cyclic projective R-module X such that $\aleph_0 X \lesssim_\oplus P$.*

Therefore, for a projective R-module P with a cyclic decomposition as above, the following conditions (e)—(h) *are equivalent*:

(e) *P is directly finite.*

(f) *P contains no infinite direct sums of nonzero pairwise isomorphic submodules.*

(g) *Every submodule of P is directly finite.*

(h) *For each nonzero cyclic projective R-module X, there exists a finite subset $\{\lambda_1, \ldots, \lambda_n\}$ of Λ such that $X \not\lesssim \bigoplus_{\lambda \in \Lambda - \{\lambda_1, \ldots, \lambda_n\}} P_\lambda$.*

Proof. It is obvious that (a) → (b) and (c) → (d) → (a) hold. (b) → (c). Assume that (b), i.e., there exists a nonzero cyclic projective R-module X such that $\aleph_0 X \lesssim P$. Let $\{\lambda_1, \ldots, \lambda_n\}$ be a subset of Λ, and set $\Lambda' = \Lambda - \{\lambda_1, \ldots, \lambda_n\}$. Since $X \lesssim P = \bigoplus_{\lambda \in \Lambda} P_\lambda$, there exists decompositions $P_\lambda = P^1_\lambda \oplus P^{(1)}_\lambda$ for each $\lambda \in \Lambda$ such that $X \simeq P^1_{\lambda_1} \oplus \cdots \oplus P^1_{\lambda_n} \oplus (\bigoplus_{\lambda \in \Lambda'} P^1_\lambda)$. Note that $2X \lesssim P = \bigoplus_{\lambda \in \Lambda} P_\lambda$ and X has the cancellation property. Then there exists decompositions $P^{(1)}_\lambda = P^2_\lambda \oplus P^{(2)}_\lambda$ for each $\lambda \in \Lambda$ such that $X \simeq P^2_{\lambda_1} \oplus \cdots \oplus P^2_{\lambda_n} \oplus (\bigoplus_{\lambda \in \Lambda'} P^2_\lambda)$. We continue this procedure. For each $m = 1, 2, \ldots$, there exist decompositions $P^{(m)}_\lambda = P^{m+1}_\lambda \oplus P^{(m+1)}_\lambda$ for each $\lambda \in \Lambda$ such that $X \simeq P^{m+1}_{\lambda_1} \oplus \ldots \oplus P^{m+1}_{\lambda_n} \oplus (\bigoplus_{\lambda \in \Lambda'} P^{m+1}_\lambda)$. Applying Lemma

1, we have a positive integer m_0 such that $X \lesssim (\bigoplus_{\lambda \in \Lambda'} P_\lambda^1) \oplus \cdots \oplus (\bigoplus_{\lambda \in \Lambda'} P_\lambda^{m_0}) <\oplus (\bigoplus_{\lambda \in \Lambda'} P_\lambda)$. Thus the condition (c) holds. The equivalences of (e)$-$(h) follows from above results.

Using the basic property (4), we obtain the following.

Lemma 3. *Let R be a regular ring of index at most n, and let I_1, $I_2, \ldots$ be a sequence of cyclic right ideals of R such that $I_i \gtrsim 2I_{i+1}$ for $i = 1, 2, \ldots$. Then we have that $I_k = 0$ for all positive integers k satisfying $2^{k-1} \geqq n+1$.*

Theorem 4. *Let R be a regular ring which has bounded index, and let k be a positive integer. If P is a directly finite projective R-module, then so is kP.*

Proof. It is sufficient to show that if P is a directly finite projective R-module then so is $2P$. Let $P = \bigoplus_{\lambda \in \Lambda} P_\lambda$ be a cyclic direct sum decomposition of P ([2]), and assume that $2P$ is directly infinite. We claim that P is directly infinite. Using Theorem 2, we have a nonzero cyclic projective R-module X such that $X \lesssim \bigoplus_{\lambda \in \Lambda - \{\lambda_1, \ldots, \lambda_n\}} 2P_\lambda$ for all finite subsets $\{\lambda_1, \ldots, \lambda_n\}$ of Λ. Hence there exist a finite subset $\{k_1, \ldots, t_1\}$ of $\Lambda - \{\lambda_1, \ldots, \lambda_n\}$ and direct summands P_i^j of P_i for $j = 1, 2$ and $i = k_1, \ldots, t_1$ such that $X \simeq (P_{k_1}^1 \oplus P_{k_1}^2) \oplus \ldots \oplus (P_{t_1}^1 \oplus P_{t_1}^2)$. Set $X_i^2 = P_i^1 \cap P_i^2$ for $i = k_1, \ldots, t_1$, and let Y_i^2 be a direct summand of P_i^2 such that $P_i^2 = X_i^2 \oplus Y_i^2$. Then $P_i^1 \oplus P_i^2 = (P_i^1 \oplus Y_i^2) \oplus X_i^2$, $P_i^1 \oplus Y_i^2 \lesssim P_i$ and $2X_i^2 \lesssim X$. Thus we have that

$$X \simeq ((P_{k_1}^1 \oplus Y_{k_1}^2) \oplus X_{k_1}^2) \oplus \cdots \oplus ((P_{t_1}^1 \oplus Y_{t_1}^2) \oplus X_{t_1}^2),$$
$$(P_{k_1}^1 \oplus Y_{k_1}^2) \oplus \cdots \oplus (P_{t_1}^1 \oplus Y_{t_1}^2) \lesssim P_{k_1} \oplus \cdots \oplus P_{t_1} \text{ and}$$
$$2X_i^2 \lesssim X \text{ for } i = k_1, \ldots, t_1.$$

Again using Theorem 2, we have a finite subset $\{k_2 = k_1^{k_1}, \ldots, t_1^{k_1}; \ldots; k_1^{t_1}, \ldots, t_1^{t_1} = t_2\}$ of $\Lambda - \{\lambda_1, \ldots, \lambda_n, k_1, \ldots, t_1\}$ and direct summands P_i^j of P_i for $i = k_2, \ldots, t_2$ and $j = 1, 2$ such that

$$X_{k_1}^2 \simeq (P_{k_1^{k_1}}^1 \oplus P_{k_1^{k_1}}^2) \oplus \cdots \oplus (P_{t_1^{k_1}}^1 \oplus P_{t_1^{k_1}}^2),$$
$$\cdots\cdots\cdots \qquad \text{and}$$
$$X_{t_1}^2 \simeq (P_{k_1^{t_1}}^1 \oplus P_{k_1^{t_1}}^2) \oplus \cdots \oplus (P_{t_1^{t_1}}^1 \oplus P_{t_1^{t_1}}^2).$$

Set $X_i^3 = P_i^1 \cap P_i^2$ for $i = k_2, \ldots, t_2$ and let Y_i^3 be a direct summand of P_i^2 such that $P_i^2 = X_i^3 \oplus Y_i^3$. Then we have that

$X^2_{k_1} \oplus \cdots \oplus X^2_{t_1} \simeq ((P^1_{k_1^{k_1}} \oplus Y^3_{k_1^{k_1}}) \oplus X^3_{k_1^{k_1}}) \oplus \cdots \oplus ((P^1_{t_1^{k_1}} \oplus Y^3_{t_1^{k_1}}) \oplus X^3_{t_1^{k_1}}) \oplus \cdots \oplus ((P^1_{k_1^{t_1}} \oplus Y^3_{k_1^{t_1}}) \oplus X^3_{k_1^{t_1}}) \oplus \cdots \oplus ((P^1_{t_1^{t_1}} \oplus Y^3_{t_1^{t_1}}) \oplus X^3_{t_1^{t_1}})$,

$(P^1_{k_1^{k_1}} \oplus Y^3_{k_1^{k_1}}) \oplus \cdots \oplus (P^1_{t_1^{t_1}} \oplus Y^3_{t_1^{t_1}}) \lesssim P_{k_2} \oplus \cdots \oplus P_{t_2}$ and $2X^3_{k_1^{k_1}}, \ldots, 2X^3_{t_1^{k_1}} \lesssim X^2_{k_1}; \ldots; 2X^3_{k_1^{t_1}}, \ldots, 2X^3_{t_1^{t_1}} \lesssim X^2_{t_1}$.

We continue this procedure. Noting that $X \lesssim R$, from Lemma 3, we have a positive integer m such that $X^m_i = 0$ for $i = k_{m-1}, \ldots, t_{m-1}$, and so $X \lesssim (P_{k_1} \oplus \cdots \oplus P_{t_1}) \oplus \cdots \oplus (P_{k_{m-1}} \oplus \cdots \oplus P_{t_{m-1}}) < \oplus\oplus_{\lambda \in \Lambda - \{\lambda_1, \ldots, \lambda_n\}} P_\lambda$. Therefore P is directly infinite from Theorem 2.

For regular rings R of bounded index, it does not hold that the direct sum of two directly finite projective R-modules is directly finite in general, as later Example shows. Therefore we shall investigate the condition for a regular ring R of bounded index that the direct sum of two directly finite projective R-modules is directly finite.

Let R be a regular ring. For a given nonzero finitely generated projective R-module P, we consider the following condition:

(#) For each nonzero finitely generated submodule Q of P and each family $\{A_1, B_1, A_2, B_2, \ldots\}$ of submodules of Q with decompositions

$$Q = A_1 \oplus B_1,$$
$$A_i = A_{2i} \oplus B_{2i} \text{ and}$$
$$B_i = A_{2i+1} \oplus B_{2i+1} \text{ for } i = 1, 2, \ldots,$$

there exists a nonzero projective R-module X such that $X \lesssim \oplus_{i=m}^{\infty} A_i$ or $X \lesssim \oplus_{i=m}^{\infty} B_i$ for each positive integer m.

Remark. 1) We can take above X as a finitely generated submodule of Q. 2) If P is a nonzero finitely generated projective R-module which satisfies the condition (#) then any nonzero direct summand of P satisfies (#).

Definition. Let P be a finitely generated projective module over a regular ring R. We use $L(P)$ to denote the lattice of all finitely generated submodules of P, partially ordered by inclusion.

Lemma 5 (cf. [1, Proposition 2.4]). *Let P be a finitely generated projective module over a regular ring R, and set $T = End_R(P)$.*

(a) *There is a lattice isomorphism ϕ: $L(T_T) \to L(P)$, given by the* *rule* $\phi(J) = JP$. *For* $A \in L(P)$, *we have* $\phi^{-1}(A) = \{f \in T \mid fP \leqq A\}$.

(b) *For* $J, K \in L(T_T)$, *we have* $J \simeq K$ *if and only if* $\phi(J) \simeq \phi(K)$.

(c) *For* $J, K \in L(T_T)$, *we have* $J \lesssim K$ *if and only if* $\phi(J) \lesssim \phi(K)$.

(d) *For* $J, K \in L(T_T)$ *such that* $J \oplus K \in L(T_T)$, *we have that* $\phi(J \oplus K) = \phi(J) \oplus \phi(K)$. *For* $A, B \in L(P)$ *such that* $A \oplus B \in L(P)$, *we have that* $\phi^{-1}(A \oplus B) = \phi^{-1}(A) \oplus \phi^{-1}(B)$.

The following is immediate from above Lemma 5.

Lemma 6. *Let* P *be a nonzero finitely generated projective module over a regular ring* R, *and set* $T = End_R(P)$. *Then the following are equivalent*:

(a) P *satisfies the condition* (#).

(b) T *satisfies the condition* (#) *as* T-*module*.

Lemma 7. *Let* R *be a regular ring. Then the following are equivalent*:

(a) R *satisfies the condition* (#) *as* R-*module*.

(b) *All nonzero finitely generated projective* R-*modules satisfy the condition* (#).

(c) *For any positive integer* k, kR *satisfies the condition* (#).

(d) *There exists a positive integer* k *such that* kR *satisfies the condition* (#).

Proof. It is obvious that (c) $\to$ (b) $\to$ (d) $\to$ (a) holds. (a) $\to$ (c). Assume that (a), and we shall prove (c) by the induction on k. Let $k > 1$ and assume that (c) holds for any positive integer smaller than k. Let Q be a nonzero finitely generated submodule of $kR = (k-1)R \oplus R$ and $\{A_1, B_1, A_2, B_2, \ldots\}$ a family of submodules of Q with decompositions $Q = A_1 \oplus B_1$, $A_i = A_{2i} \oplus B_{2i}$ and $B_i = A_{2i+1} \oplus B_{2i+1}$ for $i = 1, 2, \cdots$. Then there exist families $\{Q_1, A_1^1, B_1^1, \ldots\}$ of submodules of $(k-1)R$ and $\{Q_2, A_1^2, B_1^2, \ldots\}$ of submodules of R with decompositions

$$Q_j = A_1^j \oplus B_1^j,\ A_i^j = A_{2i}^j \oplus B_{2i}^j \text{ and } B_i^j = A_{2i+1}^j \oplus B_{2i+1}^j$$

for $i = 1, 2, \ldots$ and $j = 1, 2$ such that

$$Q \simeq Q_1 \oplus Q_2,\ A_i \simeq A_i^1 \oplus A_i^2 \text{ and } B_i \simeq B_i^1 \oplus B_i^2.$$

Since Q is nonzero, either Q_1 or Q_2 is nonzero. Now assume that Q_j is nonzero for $j = 1$ or 2. Then, from the induction hypothesis, we have a nonzero projective R-module X such that $X \lesssim \oplus_{i=m}^{\infty} A_i^j \lesssim_\oplus \oplus_{i=m}^{\infty} A_i$ or $X \lesssim \oplus_{i=m}^{\infty} B_i^j \lesssim_\oplus \oplus_{i=m}^{\infty} B_i$ for each positive integer m. As a result, kR satisfies the condition (♯), and so the proof of the lemma is complete.

For a given regular ring R, we consider the following property:

(DF) The direct sum of two directly finite projective R-modules is directly finite.

Theorem 8. *Let R be a regular ring of bounded index. Then the following are equivalent:*

(a) *R has the property* (DF).

(b) *R satisfies the condition* (♯) *as R-module.*

(c) *For any nonzero finitely generated projective R-module P, $End_R(P)$ has the property* (DF).

(d) *For any positive integer k, $M_k(R)$ has the property* (DF).

(e) *There exists a positive integer k such that $M_k(R)$ satisfies the property* (DF).

Proof. (a) → (b). Assume (a) and that the R-module R does not satisfy (♯). We have a nonzero finitely generated submodule Q of R and a family $\{A_1, B_1, A_2, B_2, \ldots\}$ of submodules of Q with decompositions $Q = A_1 \oplus B_1$, $A_i = A_{2i} \oplus B_{2i}$ and $B_i = A_{2i+1} \oplus B_{2i+1}$ for $i = 1, 2, \ldots$ such that, for each nonzero projective R-module X, there exists a positive integer m such that $X \not\lesssim \oplus_{i=m}^{\infty} A_i$ and $X \not\lesssim \oplus_{i=m}^{\infty} B_i$. Then it follows that $\oplus_{i=1}^{\infty} A_i$ and $\oplus_{i=1}^{\infty} B_i$ are directly finite from Theorem 2, but $(\oplus_{i=1}^{\infty} A_i) \oplus (\oplus_{i=1}^{\infty} B_i)$ $(\simeq \aleph_0 Q)$ is directly infinite, a contradiction. (b) → (a). We assume (b) and that (a) does not hold, i.e., there exist directly finite projective R-modules P and Q such that $P \oplus Q$ is directly infinite. According to Theorem 2, we have a nonzero cyclic right ideal X of R such that $P \oplus Q \gtrsim \aleph_0 X$. Since R is unit-regular, we can assume, without loss of generality, that P and Q are countably generated, non-finitely generated, projective R-modules. Let $P = \oplus_{i=1}^{\infty} P_i$ and $Q = \oplus_{i=1}^{\infty} Q_i$ be cyclic decompositions of P and Q. From the proof of Theorem 2, there exists a positive integer m_1 such that $\oplus_{i=m_1}^{\infty}(P_i \oplus Q_i) \gtrsim X$, consequently, there exist a positive integer s_1 $(> m_1)$ and a decomposition $X = A_1 \oplus B_1$ such that

$$A_1 \lesssim_\oplus \oplus_{i=m_1}^{s_1} P_i \text{ and } B_1 \lesssim_\oplus \oplus_{i=m_1}^{s_1} Q_i.$$

Since $A_1 \lesssim \oplus_{i=s_1+1}^{\infty}(P_i \oplus Q_i)$, there exist positive integers m_2 and s_2 satisfying $s_1 < m_2 < s_2$ and a decomposition $A_1 = A_2 \oplus B_2$ such that

$$A_2 \lesssim_\oplus \oplus_{i=m_2}^{s_2} P_i \text{ and } B_2 \lesssim_\oplus \oplus_{i=m_2}^{s_2} Q_i.$$

Next, we note that $B_2 \lesssim \oplus_{i=s_2+1}^{\infty}(P_i \oplus Q_i)$. Then there exist positive integers m_3 and s_3 satisfying $s_2 < m_3 < s_3$ and a decomposition $B_1 = A_3 \oplus B_3$ such that

$$A_3 \lesssim_\oplus \oplus_{i=m_3}^{s_3} P_i \text{ and } B_3 \lesssim_\oplus \oplus_{i=m_3}^{s_3} Q_i.$$

Continuing this procedure, we have a family $\{A_1, B_1, A_2, B_2, \ldots\}$ of cyclic submodules of X with decompositions

$$X = A_1 \oplus B_1,\ A_i = A_{2i} \oplus B_{2i} \text{ and } B_i = A_{2i+1} \oplus B_{2i-1}$$

for $i = 1, 2, \ldots$ such that

$$\oplus_{i=1}^{\infty} A_i \lesssim_\oplus P \text{ and } \oplus_{i=1}^{\infty} B_i \lesssim_\oplus Q.$$

Since P and Q are directly finite projective R-modules, so are $\oplus_{i=1}^{\infty} A_i$ and $\oplus_{i=1}^{\infty} B_i$. Hence Theorem 2 says that, for each nonzero cyclic projective R-module Y, there exists a positive integer m such that $Y \not\lesssim \oplus_{i=m}^{\infty} A_i$ and $Y \lesssim \oplus_{i=m}^{\infty} B_i$, which contradicts (b). Therefore (b) $\to$ (a) is proved. The remainder implications follow from (a) $\leftrightarrow$ (b), Lemmas 6 and 7.

Corollary 9. *The property* (DF) *for regular rings of bounded index is Morita invarient.*

Corollary 10. *If R is a regular ring of bounded index with the nonzero essential socle of R, then R has the property* (DF).

Example. *There exists a regular ring of bounded index which does not have the property* (DF).

Proof. Choose a field F, and set $R_{2^n} = \oplus_{i=1}^{2^n} F_i$, where $F_i = F$ for each i. Map each $R_{2^{n-1}} \to R_{2^n}$, given by the rule $x \to (x, x)$, and set $R = \varinjlim R_{2^n}$. For each $x \in R_{2^n}$, let $\bar{x}$ be the class of x in R. Then R is a non-artinian regular ring which has index 1 with the zero socle. Set $a_1 = (1, 0) \in R_2$, $b_1 = (0, 1) \in R_2$ and define $\{a_i, b_i\}_{i=2}^{\infty}$ inductively as the following:

$$a_{2i} = (a_i, 0) \in R_{2^{n+1}},\ b_{2i} = (0,\ a_i) \in R_{2^{n-1}} \text{ for } a_i \in R_{2^n}$$

and

$$a_{2i+1} = (b_i, 0) \in R_{2^{n+1}},\ b_{2i+1} = (0, b_i) \in R_{2^{n+1}} \text{ for } b_i \in R_{2^n}$$

Set $A_i = \bar{a}_i R$ and $B_i = \bar{b}_i R$. Then $\{A_i, B_i\}_{i=1}^{\infty}$ is a family of submodules of R with decompositions

$$R = A_1 \oplus B_1,\ A_i = A_{2i} \oplus B_{2i} \text{ and } B_i = A_{2i+1} \oplus B_{2i+1}$$

for $i = 1, 2, \ldots$. Noting that R is abelian, we have that, for each nonzero cyclic projective R-module X, $X \not\lesssim \oplus_{i=m}^{\infty} A_i$ and $X \not\lesssim \oplus_{i=m}^{\infty} B_i$ for some positive integers m. Consequently, R does not satisfy the condition (♯), and so R does not have the property (DF) from Theorem 8.

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