Math. J. Okayama Univ. 30 (1988), 41–44

A COMMUTATIVITY THEOREM FOR SEMIPRIME RINGS WITH CONSTRAINTS INVOLVING A DERIVATION

ANDRZEJ TRZEPIZUR

Throughout the present paper, R will represent a ring with center C, $d: x \to x'$ a non-trivial derivation of R, and $K = \{x \in R \mid x' = 0\}$. Let U be a non-zero ideal of R, and V the differential ideal generated by U'. As usual, we write $[x, y] = xy - yx$ and $(x, y) = xy + yx$ $(x, y \in R)$. Given subsets A and B of R, we denote by $[A, B]$ (resp. (A, B)) the additive subgroup generated by $\{[a, b] \mid a \in A,\ b \in B\}$ (resp. $\{(a, b) \mid a \in A,\ b \in B\}$).

We consider the following conditions:

a) R is commutative.

a)* R is a commutative ring of characteristic 2.

b) d is *commuting* on U, namely $[u, u'] = 0$ for every $u \in U$.

c) d is *skew-commuting* on U, namely $(u, u') = 0$ for every $u \in U$.

d) d is *centralizing* on U, namely $[u, u'] \in C$ for every $u \in U$.

e) d is *skew-centralizing* on U, namely $(u, u') \in C$ for every $u \in U$.

In case R is a prime ring, Posner [7] proved that if d is centralizing on R then R is commutative. Recently, this theorem has been generalized as follows: d) or e) implies a) (Mayne [5, 6] and Hirano-Kaya-Tominaga [1]). Furthermore, in case R is a semiprime ring, Hongan and the present author [4] have proved the following: Let U be a differential ideal of R whose left annihilator $\ell(U)$ is zero. If $K_0 = \{x \in R \mid (RxR)' = 0\}$ is commutative, then d) implies a).

The purpose of this paper is to prove the following:

Theorem 1. *Let R be a d-semiprime ring, and U a differential ideal such that $\ell(V) = 0$.*

(1) a) *and* a*) *are equivalent to* b) *and* c), *respectively, and* e) *implies* d).

(2) *If R is 2-torsion free then* a), b) *and* e) *are equivalent.*

As for definitions and fundamental results used in this paper without mention, we refer to [2, 3 and 4]. In advance of proving Theorem 1, we state several lemmas. The proof of the first one is easy, and may be omitted.

Lemma 1. *Let R be a d-prime ring. Let a be a non-zero central element of R, and b an element of R. If $a^{(k)}b \in C$ for all $k \geq 0$, then $b \in C$.*

Lemma 2. *Let R be a d-prime ring, and U a differential ideal of R.*
(1) $[U, U] = 0$ *implies* a).
(2) a) *and* b) *are equivalent.*
(3) *If R is of characteristic not* 2, *then $(U, U) \subseteq C$ implies* a).

Proof. (1) is easy and (2) is clear by [2, Lemma 7].

(3) Let s be an arbitrary element of $C \cap U$. Then, for any $u \in U$, $2s^{(k)}u = (s^{(k)}, u) \in C$, namely $s^{(k)}u \in C (k \geq 0)$. Thus, in case $C \cap U \neq 0$, Lemma 1 shows that $U \subseteq C$, so that R is commutative by (1). On the other hand, in case $C \cap U = 0$, we have $(U, U) \subseteq C \cap U = 0$. Hence, for any $u, v \in U$ and $x \in R$, we have $u^{(k)}[x, v] = (u^{(k)}x, v) - (u^{(k)}, v)x = 0$. Hence $U \subseteq C$, and therefore R is commutative.

Lemma 3. e) *implies that* $(U, U)' \subseteq C$.

Proof. By linearization of the relation $(u, u') \in C$ on U.

Lemma 4. *Let R be a d-prime ring of characteristic not* 2, *and U a differential ideal. Then* a), b) *and* e) *are equivalent.*

Proof. Since a) and b) are equivalent by Lemma 2 (2), it suffices to show that e) implies b). Let u, v be arbitrary element of U. First, suppose that $C \not\subseteq K$, and choose $c \in C$ with $c' \neq 0$. Then, by Lemma 3, $c'(u, v) = (u, cv)' - c(u, v)' \in C$, and so $c^{(k)}(u, v) \in C$ for all $k \geq 1$. Hence $(U, U) \subseteq C$ (Lemma 1), and therefore R is commutative by Lemma 2 (3). Next, suppose that $C \cap U = 0$. Then, by Lemma 3, $(U, U)' \subseteq C \cap U = 0$. Since $[(u, v), x]' = \{(u, [v, x]) + (v, [u, x])\}' = 0$ for every $x \in R$, [3, Theorem 1] proves that $(u, v) \in C$, namely $(U, U) \subseteq C$. Hence R is commutative again by Lemma 2 (3). Finally, suppose that $C \subseteq K$ and $C \cap U \neq 0$. Then, for any non-zero $s \in C \cap U$ we have $2su' = (s, u)' \in C$ (Lemma 3), and therefore $s^{(k)}u' \in C$ for all $k \geq 0$. Hence $U' \subseteq C$ (Lemma 1). Obviously, this implies b).

Corollary 1. *Let R be a d-prime ring, and U a differential ideal.*
(1) a) $\Leftrightarrow$ b) $\Rightarrow$ e) $\Rightarrow$ d).

(2) a)* $\Leftrightarrow$ c).

Proof. (1) By Lemma 2 (2) and Lemma 4.

(2) In view of Lemma 2 (2), it suffices to show that if c) is satisfied then R is of characteristic 2. If not, R is commutative by Lemma 4, and so $uu' = 0$ for every $u \in U$. Linearizing this identity, we get $(U^2)' = 0$. But this is impossible.

Proof of Theorem 1. (1) We can find a set $\{P_\lambda\}_{\lambda \in \Lambda}$ of d-prime ideals of R such that $\cap_{\lambda \in \Lambda} P_\lambda = 0$ and $U' \not\subseteq P_\lambda$ for every $\lambda \in \Lambda$. Applying Corollary 1 to $R_\lambda = R/P_\lambda$ and $U_\lambda = (U+P_\lambda)/P_\lambda$, we get the assertions.

(2) By the proof of [3, Theorem 2], we can find a subset Λ_1 of Λ such that $\cap_{\lambda \in \Lambda_1} P_\lambda = 0$ and R_λ is of characteristic not 2 for every $\lambda \in \Lambda_1$. Now, Lemma 4 proves that a), b) and e) are equivalent.

Corollary 2. *Let R be a semiprime ring, and U a differential ideal such that $\ell(V) = 0$. Then* a), b), d) *and* e) *are equivalent, and* a)* *and* c) *are equivalent.*

Proof. In view of Theorem 1, it suffices to show that d) implies b). By [4, Lemma 2], we can easily see that $[u, u']^2 = 0$ for every $u \in U$. Since R is semiprime and $[u, u'] \in C$ by d), we get $[u, u'] = 0$.

Acknowledgement. The author expresses his gratitude to the referee for the final form of this paper.

References

[1] Y. Hirano, A. Kaya and H. Tominaga: On a theorem of Mayne, Math. J. Okayama Univ. 25 (1983), 125–132.

[2] Y. Hirano and H. Tominaga: Some commutativity theorem for prime rings with derivations and differentially semiprime rings, Math. J. Okayama Univ. 26 (1984), 101–108.

[3] Y. Hirano, H. Tominaga and A. Trzepizur: On a theorem of Posner, Math. J. Okayama Univ. 27 (1985), 25–32.

[4] M. Hongan and A. Trzepizur: On generalization of a theorem of Posner, Math. J. Okayama Univ. 27 (1985), 19–23.

[5] J. H. Mayne: Ideals and centralizing mappings in prime rings, Proc. Amer. Math. Soc. 86 (1982), 211–212; Erratum, ibid. 89 (1983), 187.

[6] J. H. Mayne: Centralizing mappings of prime rings, Canad. Math. Bull. 27 (1984), 122–126.

[7] E. C. Posner: Derivations in prime rings, Proc. Amer. Math. Soc. 8 (1957), 1093–1100.

Jagellonian University
ul. Reymonta 4, Cracow, Poland

(*Received January 21, 1987*)