Math. J. Okayama Univ. 30 (1988), 25−31

ON RINGS SATISFYING THE IDENTITY $X^{2^k}=X^k$

Y. HIRANO, H. KOMATSU, H. TOMINAGA and A. YAQUB

Throughout the present paper, R will represent a ring, E the set of idempotents in R, and N the set of nilpotents in R. Our present objective is to give the conditions for R to satisfy the identity $x^{2k} = x^k$ and to reprove all the results obtained in the previous paper [5], without the extra hypothesis that R is left s-unital.

First, careful scrutiny of the proof of [1, Lemma 1] shows the next

Lemma 1. *Let m and q be positive integers, and let $k = q^{mq^m}$. Suppose that R satisfies the identity $f(x) = 0$, where $f(t)$ is a co-monic polynomial in $t\mathbf{Z}[t]$ with degree $\leq m$. If $qR = 0$ then R satisfies the identity $x^{k+k!} = x^k$, and therefore $x^{2\cdot k!} = x^{k!}$.*

Next, we shall prove

Lemma 2. *Suppose that R satisfies the identity $f(x) = 0$, where $f(t)$ is a primitive polynomial in $t\mathbf{Z}[t]$. Then there exist positive integers q and h such that $(qr)^h = 0$ for all $r \in R$.*

Proof. Consider the direct product $S = R^R$, which satisfies the same identity $f(x) = 0$. In case S coincides with its prime radical $P(S)$, R is a nil ring of bounded index. In what follows, we assume that S contains a proper prime ideal P, and choose an integer n_0 such that $q = |f(n_0)| > 0$. By [2, Theorem 7 (6)], the classical quotient ring of S/P is an Artinian simple ring satisfying the same identity $f(x) = 0$. Hence $qS \subseteq P$, which proves that $qS \subseteq P(S)$. Thus we can find a positive integer h such that $(qr)^h = 0$ for all $r \in R$.

Corollary 1. *Suppose that R satisfies the identity $f(x) = 0$, where $f(t)$ is a co-monic polynomial in $t\mathbf{Z}[t]$. Then R satisfies the identity $x^{2k} = x^k$ for some positive integer k.*

Proof. In view of Lemma 2, there exist positive integers q and h such that $(qr)^h = 0$ for all $r \in R$. Let T be the subring of R generated by $\{r^h \mid r \in R\}$. Then T satisfies the identity $f(x) = 0$ and $q^h T = 0$. Hence, by Lemma 1, there exists a positive integer k such that $r^{2kh} = r^{kh}$ for all $r \in R$.

Now, we can prove our first theorem.

Theorem 1. *The following conditions are equivalent:*

1) *There exists a primitive polynomial $f(t)$ in $t\boldsymbol{Z}[t]$ such that R satisfies the identity $f(x) = 0$.*

2) *There exists a monic polynomial $f(t)$ in $t\boldsymbol{Z}[t]$ such that R satisfies the identity $f(x) = 0$.*

3) *There exists a co-monic polynomial $f(t)$ in $t\boldsymbol{Z}[t]$ such that R satisfies the identity $f(x) = 0$.*

4) *There exists a positive integer k such that R satisfies the identity $x^{2k} = x^k$.*

5) *$qE = 0$ for some positive integer q, and there exists a positive integer m with the following property: For every $r \in R$, there exists a co-monic polynomial $g(t)$ in $t\boldsymbol{Z}[t]$ with deg $g(t) \leq m$ such that $g(r) = 0$.*

6) *The (Jacobson) radical J of R is a nil ideal of bounded index, and there exists a positive integer k such that every primitive homomorphic image of R contains at most k elements.*

In case R contains 1, the next is equivalent to each of the above equivalent conditions:

7) *The addition of R is equationally definable in terms of the multiplication and the successor operation.*

Proof. Obviously, 4) ⇒ 2) ⇒ 1), and 4) ⇒ 3) ⇒ 1).

1) ⇒ 4). Consider the direct product $S = R^R$, which satisfies the same identity $f(x) = 0$. In case S coincides with its prime radical $P(S)$, there is nothing to prove. Thus, henceforth, we may assume that S contains a proper prime ideal P. Choose an integer n_0 such that $q = |f(n_0)| > 0$. By [2, Theorem 7 (6)], the classical quotient ring of S/P is an Artinian simple ring satisfying the same identity $f(x) = 0$. Hence the characteristic of S/P is a factor of q. Noting that $f(t)$ is primitive, we can easily see that there exists a co-monic polynomial $g(t)$ in $t\boldsymbol{Z}[t]$ with deg $g(t) \leq m =$ deg $f(t)$ such that S/P satisfies the identity $g(x) = 0$. Then, by Lemma 1, there exists a positive integer $l = l(q, m)$ such that S/P satisfies the identity $x^{2l} = x^l$. This proves that $S/P(S)$ satisfies the identity $x^{2l} = x^l$. Then there exists a positive integer h such that R satisfies the identity $(x^l - x^{2l})^h = 0$. Now, by Corollary 1, there exists a positive integer k such that R

satisfies the identity $x^{2k} = x^k$.

3) ⇒ 5). Put $q = |f(2)|$, and let $g(t) = f(t)$ for all $r \in R$.

5) ⇒ 3). Let $f(t) = \prod_p \prod_{\alpha=1}^{m} (t - t^{p^\alpha})^m$, where p ranges over all the prime factors of q. We shall show that R satisfies the identity $f(x) = 0$. Now, let r be an arbitrary element of R, and let $\langle r \rangle$ be a subdirect sum of subdirectly irreducible rings R_λ. By 5), there exists a co-monic polynomial $g(t)$ in $t\boldsymbol{Z}[t]$ with $\deg g(t) \leq m$ such that $g(r) = 0$. Let N_λ be the set of nilpotents in R_λ. Then it is easy to see that $a^m = 0$ for all $a \in N_\lambda$, and so N_λ satisfies the identity $f(x) = 0$. Now, assume that R_λ is not nil. Then, as is easily seen, R_λ is a local ring whose radical is N_λ and $R_\lambda/N_\lambda = \mathrm{GF}(p^\alpha)$ with some prime factor p of q and $\alpha \leq m$. Hence $f(r) = 0$.

4) ⇒ 6). This is an easy consequence of Kaplansky's theorem (see, e.g., [2, Theorem 1]).

6) ⇒ 3). As is easily seen, every primitive homomorphic image of R satisfies the identity $x^{2\cdot k!} = x^{k!}$, and so R/J satisfies the same. Hence R satisfies the identity $(x^{k!} - x^{2\cdot k!})^h = 0$ for some positive integer h.

The latter assertion is clear by [6, Theorem 1].

Following [7], a ring R is called a *δ-ring* if R contains a finite subset S with the following property: For every $x \in R$, there exists a $p(t) \in \boldsymbol{Z}[t]$ such that $x - x^2 p(x) \in S$. As an application of Theorem 1, we shall prove the following

Theorem 2. *Let R be a δ-ring. If there exists a positive integer q such that $|K| \leq q$ for every field K which is a homomorphic image of R, then there exists a positive integer k such that R satisfies the identity $x^{2k} = x^k$.*

In preparation for proving Theorem 2, we state the next

Lemma 3. *Suppose that R contains a finite subset S with the following property: For every $x \in R$, there exists a $p(t) \in \boldsymbol{Z}[t]$ such that $x - x^2 p(x) \in S$. Let $s = |S|$. Then there holds the following:*

(1) *R is a periodic ring and N is finite.*

(2) *There is a positive integer n such that for every $x \in R$ there exists an $f(t) \in \boldsymbol{Z}[t]$ with $x^n = x^{n+1} f(x)$, and then $|N| \leq (s!)^{(n-1)s}$.*

Proof. Let x be an arbitrary element of R. For each positive integer $i \leq s+1$, there exists $g_i(t) \in \boldsymbol{Z}[t]$ such that $x^i - x^{2i} g_i(x^i) \in S$. Then we can easily see that there exists a positive integer i' and $g(t) \in Z[t]$ such that $x^{i'} = x^{i'+1}g(x)$. Hence R is periodic by Chacron's theorem (see, e.g., [3, Theorem 1]). Now, let $a \in N$; $a^k = 0$. Choose a positive integer m such that $2^m \geq k$. By hypothesis, there exist $p_1(t), \cdots, p_m(t)$ in $Z[t]$ such that $a_1 = a - a^2 p_1(a)$ and $a_j = a^{2^{j-1}} p_{j-1}(a) - a^{2^j} p_j(a)$ are in $S \cap N$ ($j = 2, \cdots, m$). Then $a = a_1 + a_2 + \cdots + a_m$. Again by hypothesis, for each positive integer $i \leq s+1$, there exists $q_i(t) \in Z[t]$ such that $ia - a^2 q_i(a) \in S$. Then we can easily see that $(s!)a = a^2 q(a)$ with some $q(t) \in Z[t]$. This implies that $(s!)^{k-1} a = a^k (q(a))^{k-1} = 0$, and hence the additive order of every element in N is finite. Combining this with the fact that every element is a sum of elements in $S \cap N$, we see that N is finite. Now, we can choose a positive integer n such that $a^n = 0$ for all $a \in N$. Since $x - x^2 g(x) \in N$, we get $0 = (x - x^2 g(x))^n = x^n - x^{n+1} f(x)$ with some $f(t) \in Z[t]$.

Proof of Theorem 2. Let S, s and n be as in Lemma 3. If R' is an arbitrary homomorphic image of R and N' is the set of nilpotents in R', then $|N'| \leq (s!)^{(n-1)s}$ by Lemma 3. This together with the structure theorem of primitive rings shows that every primitive homomorphic image of R is either a periodic field or the full matrix ring $M_m(K)$, where $1 < m \leq n$ and K is a field with $|K| \leq (s!)^{(n-1)s}$. Hence, by Theorem 1 6), R satisfies the identity $x^{2k} = x^k$ for some positive integer k.

By the proof of Theorem 2, we can easily see the following

Corollary 2. *Let R be a δ-ring. If $R = \langle E \cup N \rangle$ and $qE = 0$ for some positive integer q, then there exists a positive integer k such that R satisfies the identity $x^{2k} = x^k$.*

Next, by making use of Theorem 1, we shall improve [5, Theorems 1 and 2].

Theorem 3. *Suppose that R satisfies the identity $f(x) = 0$, where $f(t)$ is a primitive polynomial in $tZ[t]$.*

(1) *If either R is normal or $N^* = \{x \in R \mid x^2 = 0\}$ is commutative, then N is a nil ideal and R/N satisfies the identity $x = x^{k+1}$ for some $k > 1$.*

(2) *If N is commutative then N is a commutative nil ideal and R/N satisfies the identity $x = x^{k+1}$ for some $k > 1$. If, furthermore, $[[a, x], x] = 0$ for all $a \in N$ and $x \in R$, then R is commutative.*

Proof. By Theorem 1, there exists a positive integer k such that R

satisfies the identity $x^{2k} = x^k$.

(1) If R is normal, then R satisfies the identity $[x^k, y] = 0$, and therefore [4, Proposition 2] shows that N is a nil ideal of R. On the other hand, if N^* is commutative, then [5, Lemma 2 (2)] shows that N is a nil ideal of R. Needless to say, R/N satisfies the identity $x = x^{k+1}$, in either case.

(2) The former assertion is clear by (1), and the latter is immediate by [8, Theorem 1]. (If $a \in N$ and $x \in R$, then $[a, x]^2 = [a, [a, x]x] = 0$. Hence, in [5, Theorem 2 (3)], the hypothesis (iv) implies (iii).)

Given $x \in R$, we define inductively $x^{(1)} = x$, $x^{(k)} = x^{(k-1)} \circ x$, where $x \circ y = x+y+xy$. In [5], we introduced the following conditions:

(i)$_n$ $(x+x^2+\cdots+x^n)^{(n)} = 0$ for all $x \in R$.

(*) For any $x, y \in R$, $(x+xy) \circ (y+yx) = 0$ if and only if $x = y$.

In what follows, we shall reprove [5, Theorems 3, 4 and 5] without the hypothesis that R is a left s-unital ring.

Lemma 4. *Suppose that R satisfies* (i)$_{2m}$. *Then either R is a nil ring of bounded index or there exists a positive integer q such that* $qR = 0$.

Proof. There exist positive integers q' and h such that $(q'x)^h = 0$ for all $x \in R$, by Lemma 2. If $h > 1$ then $\{(q'x)^{h-1}\}^2 = 0$, and so (i)$_{2m}$ implies that $2^m(q'x)^{h-1} = 0$; hence $(2^m q'x)^{h-1} = 0$. Repeating the same argument, we obtain eventually $2^{m(h-1)}q'x = 0$ for all $x \in R$.

Now, we can improve [5, Theorems 3 and 4] as follows:

Theorem 4. *Suppose that R satisfies* (i)$_{2m}$. *Then N is a nil ideal and* $R = R_1 \oplus R_2$, *where* R_1 *is either* 0 *or a ring of odd characteristic satisfying the identity* $x = x^{k+1}$ *for some* $k > 1$, $R_2 \supseteq N$, *and* R_2/N *is a Boolean ring. If, furthermore, R is normal and N is commutative then R is commutative.*

Proof. Take Lemma 4 into account and follow the proof of [5, Theorems 3 and 4].

Finally, we shall reprove [5, Theorem 5] without assuming that R is left s-unital.

Lemma 5. *Let* $f(t) = k_1 t + k_2 t^2 + \cdots + k_m t^m$ *be a polynomial in* $t\mathbf{Z}[t]$ *with* $(k_1, k_2) = 1$. *If N satisfies the identity* $f(x) = 0$, *then N satisfies the identities* $x^3 = 0 = k_1 x + (k_2 - k_1)x^2$.

Proof. Let a be an arbitrary element of N. To see that $a^3 = 0$, it suffices to show that if $a^n = 0$ with $n \geq 4$ then $a^{n-1} = 0$. Obviously, $0 = f(a^{n-2}) = k_1 a^{n-2}$ and $0 = a^{n-3}(k_1 a + k_2 a^2 + \cdots + k_m a^m) = k_2 a^{n-1}$. Since $(k_1, k_2) = 1$, we obtain $a^{n-1} = 0$. Hence $a^3 = 0 = k_1 a + k_2 a^2$, and therefore $k_1 a + (k_2 - k_1) a^2 = k_1 a + k_2 a^2 - (k_1 a + k_2 a^2) a = 0$.

Combining Lemma 5 with Theorem 1, we readily obtain

Corollary 3. *Let $f(t) = k_1 t + k_2 t^2 + \cdots + k_m t^m$ be a polynomial in $t\mathbf{Z}[t]$ with $(k_1, k_2) = 1$. If R satisfies the identity $f(x) = 0$, then R satisfies the identity $(x - x^k)^3 = 0$ for some $k > 1$.*

Lemma 6. *Suppose that R satisfies $(\mathrm{i})_2$. Then N is a nil ideal of R and R/N is a Boolean ring.*

Proof. Since $6x^2 + 2x^4 = (x + x^2)^{(2)} + (-x + (-x)^2)^{(2)} = 0$ and $4x + 4x^3 = (x + x^2)^{(2)} - (-x + (-x)^2)^{(2)} = 0$, we get $2x^2 - 2x^4 = (6x^2 + 2x^4) - (4x + 4x^3)x = 0$, and therefore $8x^2 = (6x^2 + 2x^4) + (2x^2 - 2x^4) = 0$. Hence $2^3 x = 8x - 2(4x + 4x^3) = -8x^3 = 0$, and therefore N is a nil ideal and R/N is a Boolean ring, by [5, Lemma 3].

Lemma 7. *If R satisfies $(*)$, then R is normal.*

Proof. The assertion has been proved in the proof of [5, Theorem 5].

We are now ready to prove the following

Theorem 5. *A ring R satisfies the condition $(*)$ if and only if 1) R is commutative and R/N is a Boolean ring, and 2) $a^{(2)} = 0$ for all $a \in N$.*

Proof. Since the "if" part has been proved in the proof of [5, Theorem 5], it remains only to prove the "only if" part. Obviously, $(*)$ implies $(\mathrm{i})_2$, and so N is a nil ideal of R and R/N is a Boolean ring, by Lemma 6. Noting that R satisfies the identity $2x + 3x^2 + 2x^3 + x^4 = (x + x^2)^{(2)} = 0$, we can conclude that $a^{(2)} = 0$ for all $a \in N$ (Lemma 5). Therefore, for any $a, b \in N$, we get $a \circ b = a \circ (a \circ b)^{(2)} \circ b = b \circ a$, which shows that N is commutative. Furthermore, R is normal by Lemma 7, and so R is commutative.

REFERENCES

[1] H. ABU-KHUZAM, H. TOMINAGA and A. YAQUB : Equational definability of addition in rings

satisfying polynomial identities, Math. J. Okayama Univ. 22 (1980), 55−57.

[2] S. A. AMITSUR : Prime rings having polynomial identities with arbitrary coefficients, Proc. London Math. Soc. (3) 17 (1967), 470−486.

[3] H. E. BELL : On commutativity of periodic rings and near-rings, Acta Math. Acad. Sci. Hungar. 36 (1980), 293−302.

[4] Y. HIRANO, Y. KOBAYASHI and H. TOMINAGA : Some polynomial identities and commutativity of s-unital rings, Math. J. Okayama Univ. 24 (1982), 7−13.

[5] Y. HIRANO, H. TOMINAGA and A. YAQUB : On rings satisfying the identity $(x+x^2+\cdots+x^n)^{(n)} = 0$, Math. J. Okayama Univ. 25 (1983), 13−18.

[6] H. KOMATSU : On the equational definability of addition in rings, Math. J. Okayama Univ. 24 (1982), 133−136.

[7] M. S. PUTCHA and A. YAQUB : A finiteness condition for rings, Math. Japonica 22 (1977), 13−20.

[8] H. TOMINAGA and A. YAQUB : Some commutativity properties for rings, Math. J. Okayama Univ. 25 (1983), 81−86.

Y. HIRANO, H. KOMATSU AND H. TOMINAGA
OKAYAMA UNIVERSITY
OKAYAMA, 700 JAPAN
A. YAQUB
UNIVERSITY OF CALIFORNIA
SANTA BARBARA, CALIFORNIA 93106, U.S.A.

(*Received April 27, 1987*)