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FIXED RINGS OF SIMPLE RINGS

Dedicated to Professor Hisao Tominaga on his 60th birthday

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Let A be a ring with identity and G a finite group of ring automorphisms of A. We denote by A^G the subring of A consisting of elements $a \in A$ such that $\sigma(a) = a$ for all $\sigma \in G$.

A need not be finitely generated over A^G even if A is a simple artinian ring as shown by Björk [1]. However, if the order of G is invertible in A, we can obtain the following result.

Theorem. *If A is a finite direct sum of simple rings and if the order of G is invertible in A, then A is a Frobenius extension of A^G.*

The purpose of this paper is to show the above result.

Throughout this paper, all rings, subrings, ring homomorphisms and modules are assumed to be unital.

According to Kasch [2], a ring extension A/B is called a *Frobenius extension* provided that A is finitely generated projective as a right B-module and $A \cong \mathrm{Hom}(A_B, B_B)$ as B-A-bimodules. As shown in Onodera [7], A/B is a Frobenius extension if and only if there exist a B-B-homomorphism h of A to B and a finite number of elements r_i's, l_i's in A such that $x = \sum_i r_i h(l_i x) = \sum_i h(xr_i) l_i$ for all $x \in A$. When this is the case, we shall call $(h;\ r_i,\ l_i)_i$ a *Frobenius system* for A/B.

The following is obvious from the definition of Frobenius extension.

Lemma 1. *Let A_i/B_i $(i = 1, \ldots, n)$ be ring extensions. Then the finite product $A_1 \times \cdots \times A_n$ of rings is a Frobenius extension of $B_1 \times \cdots \times B_n$ if and only if each A_i/B_i is a Frobenius one.*

The following is well-known (see [7]).

Lemma 2. *Let A/B be a Frobenius extension. Then*

(1) *For any Frobenius extension A'/A, A'/B is a Frobenius one.*

(2) *Suppose B is contained in the center of A. For any algebra B' over B, $A \otimes_B B'/B' (\cong B \otimes_B B')$ is a Frobenius extension.*

(3) *For any left A-module X,* $\mathrm{Hom}({}_AX, {}_AA) \cong \mathrm{Hom}({}_BX, {}_BB)$.

Lemma 3. *Let A_0/B_0 be a Frobenius extension. Let $f_j: A_0 \to A_j$ ($j = 1,\dots,m$) be ring isomorphisms. Then the finite product $A = A_0 \times A_1 \times \cdots \times A_m$ of rings is a Frobenius extension of $B = \{(x, f_1(x),\dots,f_m(x));\ x \in B_0\}$.*

Proof. Let $(h_0;\ r_i, l_i)_{1\le i\le n}$ be a Frobenius system for A_0/B_0. Let define a mapping h of A to B by

$h(x_0, f_1(x_1),\dots,f_m(x_m)) = (y, f_1(y),\dots,f_m(y))$ for $x_0, x_1,\dots,x_m \in A_0$, where $y = h_0(x_0+x_1+\cdots+x_m)$. Let $r_{j,i}$, $l_{j,i}$ ($0 \le j \le m$, $1 \le i \le n$) be the elements of A defined as follows:

$$r_{0,i} = (r_i, 0,\dots,0), \qquad l_{0,i} = (l_i, 0,\dots,0)$$

$$\dots\dots$$

$$r_{m,i} = (0,\dots,0, f_m(r_i)),\ l_{m,i} = (0,\dots,0, f_m(l_i)) \qquad (i = 1,\dots,n).$$

Then one can see that $(h;\ r_{j,i}, l_{j,i})_{j,i}$ is a Frobenius system for A/B.

A ring A is simple if it has no proper two-sided ideals. Let G be a finite group of ring automorphisms of A. G is inner if every element σ of G is inner, that is, there exists a unit u in A such that $\sigma(a) = uau^{-1}$ for all $a \in A$, and G is outer if the identity element of G is the only inner automorphism in G.

The following result is due to Miyashita [4].

Lemma 4. *If a ring A is simple and if G is outer, then A is a Frobenius extension of A^G.*

Lemma 5. *If a ring A is simple and if G is inner such that its order is invertible in A, then A is a Frobenius extension of A^G.*

Proof. Let S be the algebra of G, that is $S = \sum_{\sigma\in G} J(\sigma)$, where $J(\sigma) = \{x \in A;\ xa = \sigma(a)x$ for all a $\in A\}$. Let C be the center of A. S is then a finite dimensional separable algebra over C (see, for example, [6], page 28). Further, the centralizer of S in A coincides with A^G. Let $S = S_1 \oplus \cdots \oplus S_n$ be a decomposition of S into simple rings. Let $T = A \otimes_C S^o$, $T_i = A \otimes_C S_i^o$ ($i = 1,\dots,n$), where S^o and S_i^o denote the opposite rings of S and S_i respectively. We will show that when we consider A a left T-module by means of $(a \otimes s^o)x = axs$, A is a generator. Since S^o and S_i^o's are Frobenius extensions of C, T and T_i's are so over A by Lemma 2(2). Hence, by Lemma 2(3), $\mathrm{Hom}({}_{T_i}Ae_i, {}_{T_i}T_i) \cong \mathrm{Hom}({}_AAe_i, {}_AA) \neq 0$, where e_i denotes the identity element of S_i. Since T_i is simple, Ae_i is a generator over T_i, and so A is a generator over T as desired. Thus $T \otimes_A A$ ($\cong T$) is isomorphic to a direct summand of a finite direct sum of copies of A as a left T-module. Therefore, recalling T/A a Frobenius extension, we have by Theorem 2.10 of [5] that $\mathrm{End}({}_A A)/\mathrm{End}({}_T A)$, or equivalently, A/A^G is a Frobenius extension.

We are now in position to prove the theorem.

Proof of Theorem. We assume first A is simple, and show the theorem by induction on the order $|G|$ of G. Let N be the normal subgroup of G consisting of inner automorphisms in G. By Lemmas 4, 5, we may assume that $1 < |N| < |G|$. Let $T = A^N$, $\overline{G} = G/N$. $\overline{G}$ acts as automorphisms on T. By our induction hypothesis, A is a Frobenius extension of T. We shall show that $T/T^{\overline{G}}$ is a Frobenius extension. By [3], T is a finite direct sum of simple rings. Let $T = T_0 \oplus T_1 \oplus \cdots \oplus T_m$ be a decomposition of T into simple rings. Since every element of $\overline{G}$ induces a permutation of the finite set $\{T_0, T_1, \ldots, T_m\}$, we can assume by Lemma 1 that $\overline{G}$ is transitive on the set. Let $\overline{\sigma}_i$ be elements of $\overline{G}$ such that $\overline{\sigma}_i(T_0) = T_i (i = 1, \ldots, m)$, and let $\overline{G}_0$ be the set of $\overline{\sigma} \in \overline{G}$ such that $\overline{\sigma}(T_0) = T_0$. Then it is easy to see that $T^{\overline{G}} = \{x + \overline{\sigma}_1(x) + \cdots + \overline{\sigma}_m(x) ; x \in T_0^{\overline{G}_0}\}$. Since T_0 is a Frobenius extension of $T_0^{\overline{G}_0}$ by our induction hypothesis, T is so over $T^{\overline{G}}$ ($= A^G$) by Lemma 3. Hence A is a Frobenius extension of A^G by Lemma 2(1).

We shall next consider a general case. Let $A = A_0 \oplus A_1 \oplus \cdots \oplus A_m$ be a representation of A as a finite direct sum of simple rings. Let G_0 be the set of $\sigma \in G$ with $\sigma(A_0) = A_0$. Then A_0 is a Frobenius extension of $A_0^{G_0}$ from the first case, so that A is a Frobenius extension of A^G by the same argument as in proving T a Frobenius one over $T^{\overline{G}}$ above.

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