

## HIGHER DERIVATIONS OF ALGEBRAIC FUNCTION FIELDS

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Let  $A$  be a commutative  $k$ -algebra with unity 1 where  $k$  is a field of characteristic zero. A higher derivation on  $A$  is a sequence  $\underline{D} = \{D_0, D_1, D_2, \dots\}$  of endomorphisms of the  $k$ -module  $A$  where  $D_0$  is the identity map of  $A$  and  $D_n(ab) = \sum_{m=0}^n D_m(a)D_{n-m}(b)$  for every  $a, b \in A$ . Let  $\phi_{\underline{D}}$  be the embedding of  $A$  into  $A[t]$  the ring of formal power series in indeterminate  $t$  induced by the higher derivation  $\underline{D}$  and defined by  $\phi_{\underline{D}}(a) = \sum_{n=0}^{\infty} D_n(a)t^n$  for every  $a \in A$ . We say that two higher derivations  $\underline{D} = \{D_n: n \geq 0\}$ , and  $\underline{E} = \{E_n: n \geq 0\}$  are equivalent if and only if  $\sigma\phi_{\underline{E}} = \phi_{\underline{D}}$  for some  $A$ -automorphism  $\sigma$  of  $A[t]$ . In a previous paper [1] the author proved that there is a one to one correspondence between the set of all higher derivations on  $A$  and the set of ordered sequences of first order derivations on  $A$ . In this paper we prove that every higher derivation  $\underline{D} = \{D_n: n \geq 0, D_1 \neq 0\}$  of an algebraic function field  $K$  of transcendency degree one over a field  $k$  of characteristic zero is equivalent to the higher derivation  $\underline{\delta} = \{(1/n!)\delta^n: n \geq 0\}$  where  $\delta$  is the unique extension to  $K$  of the ordinary derivation  $d/dx$  of  $k(x)$ .

Our terminology is essentially the same as that of [1].

**Definition.** Two higher derivations  $\underline{D} = \{D_n: n \geq 0\}$  and  $\underline{E} = \{E_n: n \geq 0\}$  on  $A$  are said to be equivalent if and only if there exists an  $A$ -automorphism  $\sigma$  of  $A[t]$  such that  $\sigma\phi_{\underline{E}} = \phi_{\underline{D}}$  where  $\phi_{\underline{E}}, \phi_{\underline{D}}$  are the embeddings of  $A$  into  $A[t]$  induced respectively by  $\underline{E}$  and  $\underline{D}$ .

**Theorem 1([1]).** There is a one to one correspondence between the set of ordered sequences of  $k$ -derivations on  $A$  of order one and the set of higher derivations on  $A$  in such a way that if  $\{\delta_n: n \geq 0, \delta_0$  identity,  $\delta_n$  is a  $k$ -derivation of order one $\}$  and the higher derivation  $\underline{D} = \{D_n: n \geq 0\}$  correspond to each other, then

$$D_n = \sum_{r=1}^n \frac{1}{r!} \sum_{\{n_1, \dots, n_r\} \in P_{n,r}} \delta_{n_1} \delta_{n_2} \cdots \delta_{n_r}$$

and

$$\delta_n = \sum_{r=1}^n \frac{(-1)^{r+1}}{r} \sum_{(n_1, \dots, n_r) \in P_{n,r}} D_{n_1} D_{n_2} \cdots D_{n_r}$$

for every  $n \geq 1$  and  $D_0 = \delta_0$ , where  $P_{n,r}$  stands for the set of ordered partitions of  $n$  into  $r$ -positive integers.

**Theorem 2.** *If  $K$  is an algebraic function field of transcendence degree one over a field  $k$  of characteristic zero, then every higher derivation  $\underline{D} = \{D_n: n \geq 0, D_1 \neq 0\}$  on  $K$  is equivalent to the higher derivation  $\underline{V} = \{(1/n!) \delta^n: n \geq 0\}$  where  $\delta$  is the unique extension to  $K$  of the ordinary derivation  $d/dx$  of  $k(x)$ .*

*Proof.* By Theorem 1, let  $\{\delta_n: n \geq 0\}$  be the sequence of  $k$ -derivations on  $K$  of order one associated to  $\underline{D}$ . Hence

$$D_n = \sum_{r=1}^n \frac{1}{r!} \sum_{(n_1, \dots, n_r) \in P_{n,r}} \delta_{n_1} \delta_{n_2} \cdots \delta_{n_r}$$

but since  $K$  is a simple separable extension of  $k(x)$  we have by [2, Theorem 4.3.10]

$$\delta_{n_i} = C_{n_i} \delta \text{ for some } C_{n_i} \in K$$

then by Leibniz Rule we get for  $r \geq 1$ ,

$$\delta_{n_1} \delta_{n_2} \cdots \delta_{n_r} = \sum_{q=1}^r \left[ \sum_{(q_{0,1}, q_{0,2}, \dots, q_{0,r}) \in B_{r,q}} \times Q(q_{0,1}, q_{0,2}, \dots, q_{0,r}) C_{n_1}^{q_{0,1}} C_{n_2}^{q_{0,2}} \cdots C_{n_r}^{q_{0,r}} \right] \delta^q$$

where  $B_{r,q} = \{(q_{0,1}, q_{0,2}, \dots, q_{0,r}): q_{0,i}$ 's are non-negative integers such that  $q_{0,1} + q_{0,2} + \cdots + q_{0,r} = r - q$  and  $q_{0,1} + \cdots + q_{0,i} \leq \min(i-1, r-q)$  for every  $1 \leq i \leq r\}$  and  $Q(q_{0,1}, q_{0,2}, \dots, q_{0,r}) = \binom{0}{q_{0,1}} \binom{1-q_{0,1}}{q_{0,2}} \binom{2-q_{0,1}-q_{0,2}}{q_{0,3}} \cdots \binom{r-1-q_{0,1}-\cdots-q_{0,r-1}}{q_{0,r}}$  and  $C_{n_i}^{q_{0,i}} = \delta^{q_{0,i}}(C_{n_i})$ ,  $\delta^0(C_{n_i}) = C_{n_i}$ .

Notice that  $q_{0,1} = 0$  and if  $\lambda = (q'_{0,1}, \dots, q'_{0,m_0}, \dots, q'_{s,1}, \dots, q'_{s,m_s}, \dots, q'_{t,1}, \dots, q'_{t,m_t})$  is a permutation of  $(q_{0,1}, q_{0,2}, \dots, q_{0,r})$  such that  $\sum_{s=0}^t m_s = r$ ,  $q'_{s,j} = q'_{s,j+1}$  and  $q'_{s,j} < q'_{s+1,j}$  for every  $0 \leq s \leq t$  then also  $\lambda \in B_{r,q}$ .

Let

$$\alpha_n = \sum_{r=1}^n \frac{1}{r!} \sum_{(n_1, \dots, n_r) \in P_{n,r}} \sum_{(q_{0,1}, q_{0,2}, \dots, q_{0,r}) \in B_{r,1}} \times Q(q_{0,1}, q_{0,2}, \dots, q_{0,r}) C_{n_1}^{q_{0,1}} C_{n_2}^{q_{0,2}} \cdots C_{n_r}^{q_{0,r}}$$

for  $n \geq 1$ . Then we get

$$D_n = \alpha_n \delta + \sum_{q=2}^n \left[ \sum_{r=q}^n \frac{1}{r!} \sum_{(n_1, \dots, n_r) \in P_{n,r}} \sum_{(q_{0,1}, q_{0,2}, \dots, q_{0,r}) \in B_{r,q}} \right. \\ \left. \times Q(q_{0,1}, q_{0,2}, \dots, q_{0,r}) C_{n_1}^{q_{0,1}} C_{n_2}^{q_{0,2}} \dots C_{n_r}^{q_{0,r}} \right] \delta^q$$

for  $n \geq 2$ .

Define

$$\sigma(t) = \sum_{n=1}^{\infty} \alpha_n t^n.$$

Since  $D_1 = \delta_1 = c_1 \delta = \alpha_1 \delta \neq 0$  hence  $\alpha_1 \neq 0$ . It is easily seen that  $\sigma$  can be extended to a  $K$ -automorphism of  $K[t]$ , and that, for such  $\sigma$ , we have  $\sigma \phi_{\underline{e}} = \phi_{\underline{e}}$  if and only if

$$D_n = \sum_{q=1}^n \frac{1}{q!} \left[ \sum_{(n_1, \dots, n_q) \in P_{n,q}} \alpha_{n_1} \alpha_{n_2} \dots \alpha_{n_q} \right] \delta^q.$$

Hence to prove the assertion it is sufficient to show that

$$\sum_{r=q}^n \frac{1}{r!} \sum_{(n_1, \dots, n_r) \in P_{n,r}} \sum_{(q_{0,1}, q_{0,2}, \dots, q_{0,r}) \in B_{r,q}} \\ \times Q(q_{0,1}, q_{0,2}, \dots, q_{0,r}) C_{n_1}^{q_{0,1}} C_{n_2}^{q_{0,2}} \dots C_{n_r}^{q_{0,r}} \\ = \frac{1}{q!} \sum_{(n_1, \dots, n_q) \in P_{n,q}} \alpha_{n_1} \alpha_{n_2} \dots \alpha_{n_q} \text{ for every } q \geq 1.$$

For convenience let  $B_{i_j,1} = \{(q'_{j,1}, \dots, q'_{j,i_j}) : q'_{j,e}$ 's are non-negative integers such that  $q'_{j,1} + \dots + q'_{j,i_j} = i_j - 1$  and  $q'_{j,1} + \dots + q'_{j,e} \leq \min(e-1, i_j-1)$  for every  $1 \leq e \leq i_j\}$  and

$$Q_{i_j} = Q(q'_{j,1}, \dots, q'_{j,i_j}) \\ R_{i_j} = C_{n_{j1}}^{q'_{j,1}} C_{n_{j2}}^{q'_{j,2}} \dots C_{n_{ji_j}}^{q'_{j,i_j}}.$$

It is easily seen that for every  $q \geq 1$ ,  $n \geq 2$  we have

$$\sum_{(n_1, \dots, n_q) \in P_{n,q}} \alpha_{n_1} \alpha_{n_2} \dots \alpha_{n_q} \\ = \sum_{(n_1, \dots, n_q) \in P_{n,q}} C_{n_1} C_{n_2} \dots C_{n_q} \\ + \sum_{r=q+1}^n \sum_{(n_1, \dots, n_q) \in P_{n,q}} \sum_{(i_1, \dots, i_q) \in P_{r,q}} \\ \times \frac{1}{i_1! i_2! \dots i_q!} \sum_{\substack{(n_{j1}, \dots, n_{ji_j}) \in P_{n_j, i_j} \\ (q'_{j,1}, \dots, q'_{j,i_j}) \in B_{i_j,1} \\ \text{for every } 1 \leq j \leq q}} (Q_{i_1} \dots Q_{i_q} R_{i_1} \dots R_{i_q}).$$

Hence it is sufficient to show that

$$\begin{aligned} & \frac{1}{r!} \sum_{(n_1, \dots, n_r) \in P_{n,r}} \sum_{(q_{0,1}, \dots, q_{0,r}) \in B_{r,q}} \\ & \quad \times Q(q_{0,1}, \dots, q_{0,r}) C_{n_1}^{q_{0,1}} C_{n_2}^{q_{0,2}} \dots C_{n_r}^{q_{0,r}} \\ & = \frac{1}{q!} \sum_{(n_1, \dots, n_q) \in P_{n,q}} \sum_{(i_1, \dots, i_q) \in P_{r,q}} \\ & \quad \times \sum_{\substack{(n_{j_1}, \dots, n_{j_i}) \in P_{n,j,i} \\ (q'_{j_1,1}, \dots, q'_{j_i,i_j}) \in B_{i,j,1} \\ \text{for every } 1 \leq j \leq q}} \left( \frac{1}{i_1! \dots i_q!} Q_{i_1} \dots Q_{i_q} R_{i_1} \dots R_{i_q} \right) \end{aligned}$$

for every  $1 \leq q \leq r \leq n$ .

It is clear that both sides of this relation have the same number of terms and that they are equal for  $q = 1$ ,  $r \geq 1$  and for  $q = r$ .

Let

$$\lambda = (q_{0,1}, \dots, q_{0,m_0}, \dots, q_{s,1}, \dots, q_{s,m_s}, \dots, q_{t,1}, \dots, q_{t,m_t}) \in B_{r,q}$$

such that  $\sum_{s=0}^t m_s = r$ ,  $q_{s,j} = q_{s,j+1}$  and  $q_{s,j} < q_{s+1,j}$  for every  $0 \leq s \leq t$ , and

$$[\lambda] = \{ \mu \in B_{r,q} : \mu \text{ is a permutation of } \lambda \}.$$

Then for any  $(n_{0,1}, \dots, n_{t,m_t}) \in P_{n,r}$  it is seen without essential difficulty that the coefficient of  $C_{n_{0,1}}^{q_{0,1}} C_{n_{0,2}}^{q_{0,2}} \dots C_{n_{t,m_t}}^{q_{t,m_t}}$  in the left side of this relation after collecting similar terms is

$$\frac{1}{r!} \sum_{\mu \in [\lambda]} L_1 L_2 \dots L_t Q(\mu)$$

where  $L_s$  is the number of permutations of  $(n_{s,1}, \dots, n_{s,m_s})$  for every  $0 \leq s \leq t$ , and the coefficient of the same expression in the right side of this relation is

$$\frac{1}{q!} \left[ \sum_{\substack{(q'_{j_1,1}, \dots, q'_{j_i,i_j}) \in B_{i,j,1} \\ \text{for every } 1 \leq j \leq q \text{ such that} \\ (q'_{1,1}, \dots, q'_{1,i_1}, \dots, q'_{q,1}, \dots, q'_{q,i_q}) \in [\lambda] \\ \text{and } (i_1, \dots, i_q) \in P_{r,q}}} \left( L_1 L_2 \dots L_t \cdot \frac{Q_{i_1} Q_{i_2} \dots Q_{i_q}}{(i_1)! (i_2)! \dots (i_q)!} \right) \right].$$

Let  $\mu \in [\lambda]$  which can be written in the form  $\mu = (\mu_{i_1}, \dots, \mu_{i_p})$  for some positive integers,  $i_1 < i_2 < \dots < i_p$  where  $\mu_{i_j} = (\lambda_{1,i_j}^j \text{ repeated } n_{j_1} \text{ times}, \dots, \lambda_{s_j,i_j}^j \text{ repeated } n_{j_s_j} \text{ times})$  and

$$\lambda_{k,i_j}^j = (q_{k,1}^j, \dots, q_{k,i_j}^j) \in B_{i_j,1} \text{ for every } 1 \leq j \leq p \text{ and } 1 \leq k \leq s_j$$

such that  $\sum_{j=1}^p \sum_{k=1}^{s_j} n_{jk} = q$ ,  $\sum_{j=1}^p [\sum_{k=1}^{s_j} n_{jk}] i_j = r$  and  $\lambda_{k_1, i_j}^j \neq \lambda_{k_2, i_j}^j$  for every  $1 \leq j \leq p$  and  $1 \leq k_1 \neq k_2 \leq s_j$  and such that ' $q_{k,e}^j$ 's in each  $\lambda_{k, i_j}^j$  are ordered in the same way as in  $\lambda$ .

Let  $\bar{\mu} = |\mu' \in [\lambda]|$ :  $\mu'$  is obtained from  $\mu$  by a permutation of  $\lambda_{k, i_j}^j$ 's in  $\mu$ .

It is clear that such a  $\mu$  exists and  $|\bar{\mu}| = \frac{q!}{\prod_{j=1}^p \prod_{k=1}^{s_j} (n_{jk})!}$ . For simplicity we set

$$f(r, q; \lambda) = \sum_{\mu \in |\lambda|} Q(\mu)$$

and

$$N_j = \sum_{k=1}^{s_j} n_{jk}$$

and

$$g(r, q; \lambda) = \frac{r!}{q!} \sum_{\substack{\{q_{j,1}^j, \dots, q_{j,i_j}^j\} \in B_{i_j,1} \\ \text{for every } 1 \leq j \leq q \\ \text{such that} \\ \{q_{1,1}^1, \dots, q_{1,i_1}^1, \dots, q_{q,i_q}^q\} \in |\lambda|}} \left( \frac{1}{(i_1)! \dots (i_p)!} \cdot Q_{i_1} Q_{i_2} \dots Q_{i_q} \right).$$

Then it is easily seen that

$$g(r, q; \lambda) = \frac{r!}{q!} \sum_{\substack{\mu \in |\lambda| \\ \text{corresponding to} \\ \text{distinct classes } \bar{\mu}}} \left[ \frac{1}{(i_1!)^{N_1} \dots (i_p!)^{N_p}} \cdot \frac{q!}{\prod_{j=1}^p \prod_{k=1}^{s_j} (n_{jk})!} \cdot \prod_{k=1}^p \prod_{k=1}^{s_j} (f(i_{j,1}; \lambda_{k, i_j}^j))^{n_{jk}} \right].$$

Hence to prove the assertion it is sufficient to show that  $f(r, q; \lambda) = g(r, q; \lambda)$  for every  $r \geq q \geq 1$  and  $\lambda \in B_{r,q}$ .

We prove this equality by induction on  $r$ .

(1) It is clear that the equality holds for  $q = 1$ ,  $r \geq 1$  and for  $q = r$  and that  $f(r, r; \lambda) = g(r, r; \lambda) = 1$ .

(2) We prove the following two lemmas.

**Lemma 1.** *Let  $\lambda_s \in B_{r-1, q_{s,1}+q-1}$  obtained by deleting  $q_{s,1}$  from  $\lambda$ , then we have*

$$f(r, q; \lambda) = \sum_{s=0}^t \binom{q_{s,1}+q-1}{q_{s,1}} f(r-1, q_{s,1}+q-1; \lambda_s)$$

for every  $q \geq 2$  and  $f(r, 1; \lambda) = \sum_{s=1}^t f(r-1, q_{s,1}; \lambda_s)$  for  $r > 1$ .

*Proof.* It follows easily from the definition of  $Q(\lambda_s)$ .

**Lemma 2.** Let  $\gamma^e = (q_{0,1}^e, \dots, q_{0,h_0}^e, \dots, q_{s',1}^e, \dots, q_{s',h_{s'}}^e, \dots, q_{i,1}^e, \dots, q_{i,h_i}^e) \in B_{e,q-1}$  and  $\gamma^{r-e} = (q_{0,1}^{r-e}, \dots, q_{0,z_0}^{r-e}, \dots, q_{s',1}^{r-e}, \dots, q_{s',z_{s'}}^{r-e}, \dots, q_{\nu,1}^{r-e}, \dots, q_{\nu,z_\nu}^{r-e}) \in B_{r-e,1}$  where  $r > e \geq q-1$  and the  $q_{0,i}^e$ 's in  $\gamma^e$  and  $q_{0,i}^{r-e}$ 's in  $\gamma^{r-e}$  are ordered in the same way as in  $\lambda$  and such that  $(\gamma^e, \gamma^{r-e}) = (q_{0,1}^e, \dots, q_{i,h_i}^e, q_{0,1}^{r-e}, \dots, q_{\nu,z_\nu}^{r-e}) \in [\lambda]$ . Then we have

$$g(r, q; \lambda) = \sum_{\substack{(e, r-e) \in P_{r,2} \\ \text{such that there exist} \\ \gamma^e \in B_{e,q-1} \text{ and} \\ \gamma^{r-e} \in B_{r-e,1} \text{ with } (\gamma^e, \gamma^{r-e}) \in [\lambda]}} \times \sum_{\substack{\gamma^e \in B_{e,q-1} \\ \gamma^{r-e} \in B_{r-e,1} \\ \text{such that} \\ (\gamma^e, \gamma^{r-e}) \in [\lambda]}} \binom{r-1}{e} g(e, q-1; \gamma^e) f(r-e, 1; \gamma^{r-e}).$$

*Proof.* Notice that the expression  $\prod_{j=1}^p \prod_{k=1}^{s_j} [f(i_j, 1; \lambda_{k,i_j}^j)]^{n_{jk}}$  appears in the right side when  $r-e = i_j$  and  $\gamma^{r-e} = \lambda_{k,i_j}^j$  and  $\gamma^e$  is obtained from  $\mu$  by deleting  $\lambda_{k,i_j}^j$  for every  $1 \leq j \leq p$  and  $1 \leq k \leq s_j$ . Hence its coefficient is

$$\begin{aligned} & \sum_{j=1}^p \frac{(r-1)!}{(r-i_j)!(i_j-1)!} \cdot \frac{(r-i_j)!}{(q-1)!} \cdot \frac{(i_j)!}{(i_1!)^{N_1} \dots (i_p!)^{N_p}} \cdot \frac{\sum_{k=1}^{s_j} n_{jk} \cdot (q-1)!}{\prod_{j=1}^p \prod_{k=1}^{s_j} (n_{jk})!} \\ &= \frac{(r-1)!}{(i_1!)^{N_1} \dots (i_p!)^{N_p} \prod_{j=1}^p \prod_{k=1}^{s_j} (n_{jk})!} \cdot \sum_{j=1}^p [\sum_{k=1}^{s_j} n_{jk}] i_j \\ &= \frac{r!}{(i_1!)^{N_1} \dots (i_p!)^{N_p} \prod_{j=1}^p \prod_{k=1}^{s_j} (n_{jk})!} \end{aligned}$$

= the coefficient of the same expression in  $g(r, q; \lambda)$ .

(3) By Lemma 1 and induction hypothesis we have

$$f(r, q; \lambda) = \sum_{s=0}^t \binom{q_{s,1} + q - 1}{q_{s,1}} g(r-1, q_{s,1} + q - 1; \lambda_s).$$

On the other hand by Lemma 2 we have

$$g(r, q; \lambda) = \sum_{\substack{(e, r-e) \in P_{r,2} \\ \text{such that there exist} \\ \gamma^e \in B_{e,q-1} \text{ and} \\ \gamma^{r-e} \in B_{r-e,1} \text{ with } (\gamma^e, \gamma^{r-e}) \in [\lambda]}} \times \sum_{\substack{\gamma^e \in B_{e,q-1} \\ \gamma^{r-e} \in B_{r-e,1} \\ \text{such that} \\ (\gamma^e, \gamma^{r-e}) \in [\lambda]}} \binom{r-1}{e} g(e, q-1; \gamma^e) f(r-e, 1; \gamma^{r-e}).$$

Since the term in  $f(r, q; \lambda)$  corresponding to  $s = 0$  is equal to the term in  $g(r, q; \lambda)$  corresponding to  $r - e = 1$  and by Lemma 2 and induction hypothesis we have

$$f(r - e, 1; \gamma^{r-e}) = \sum_{s'=1}^{\nu} g(r - e - 1, q_{s',1}^{r-e}; \gamma_{s',1}^{r-e}) \text{ for } r - e > 1$$

where  $\gamma_{s',1}^{r-e}$  is obtained from  $\gamma^{r-e}$  by deleting  $q_{s',1}^{r-e}$ .

Hence it is sufficient to show that

$$\begin{aligned} & \sum_{s=1}^t \binom{q_{s,1} + q - 1}{q_{s,1}} g(r - 1, q_{s,1} + q - 1; \lambda_s) \\ &= \sum_{\substack{i: (e, r-e-1) \in P_{r-1,2} \\ \text{such that there exist } \gamma^e \in B_{e, q-1} \text{ and} \\ \gamma_{s',1}^{r-e} \in B_{r-e-1, q_{s',1}^{r-e}} \text{ with } (\gamma^e, \gamma^{r-e}) \in |\lambda|}} \\ & \times \sum_{\substack{\gamma^e \in B_{e, q-1} \\ \gamma_{s',1}^{r-e} \in B_{r-e-1, q_{s',1}^{r-e}} \\ \text{such that } (\gamma^e, \gamma^{r-e}) \in |\lambda|}} \left[ \sum_{s'=1}^{\nu} \binom{r-1}{e} g(e, q-1; \gamma^e) \right. \\ & \left. \cdot g(r - e - 1, q_{s',1}^{r-e}; \gamma_{s',1}^{r-e}) \right]. \end{aligned}$$

Since

$$\begin{aligned} & g(r - 1, q_{s,1} + q - 1; \lambda_s) \\ &= \frac{(r-1)!}{(q_{s,1} + q - 1)!} \sum_{\substack{\mu \in |\lambda_s| \\ \text{corresponding to} \\ \text{distinct classes } \bar{\mu}}} \frac{1}{(i_1!)^{M_1} \dots (i_{p'})^{M_{p'}}} \\ & \cdot \frac{(q_{s,1} + q - 1)!}{\prod_{j=1}^{p'} \prod_{k=1}^{s_j} (n'_{jk})!} \cdot \prod_{j=1}^{p'} \prod_{k=1}^{s_j} [f(i_j, 1; \Omega_{k,i_j}^j)]^{n_{jk}} \end{aligned}$$

where  $\sum_{j=1}^{p'} \sum_{k=1}^{s_j} n'_{jk} = q_{s,1} + q - 1$ ,  $\sum_{j=1}^{p'} [\sum_{k=1}^{s_j} n'_{jk}] i_j = r - 1$  and  $M_j = \sum_{k=1}^{s_j} n'_{jk}$ .

Let  $T$  be the collection of all factors in the expression  $\prod_{j=1}^{p'} \prod_{k=1}^{s_j} [f(i_j, 1; \Omega_{k,i_j}^j)]^{n_{jk}}$  and for each  $1 \leq s \leq t$  let  $T_{q_{s,1}}$  and  $T_{q-1}$  be a partition of  $T$  into two collections containing respectively  $q_{s,1}$  and  $q - 1$  factors. It is clear that if  $P(T)$  is the number of permutations of  $T$  then we have

$$\begin{aligned} P(T) &= \sum_{T_{q_{s,1}} \cup T_{q-1} = T} P(T_{q_{s,1}}) \cdot P(T_{q-1}) \\ &= \frac{(q_{s,1} + q - 1)!}{\prod_{j=1}^{p'} \prod_{k=1}^{s_j} (n'_{jk})!}. \end{aligned}$$

Next the assertion follows easily from the fact that for each  $1 \leq s \leq t$  the coefficient of the expression  $\prod_{j=1}^{p'} \prod_{k=1}^{s_j} [f(i_j, 1; \Omega_{k,i_j}^j)]^{n_{jk}}$  in the left side is

$$= \frac{(r-1)!}{(q-1)!(q_{s,1})!} \cdot \frac{1}{(i_1!)^{m_1} \dots (i_{p'})^{m_{p'}}} \cdot P(T).$$

While the coefficient of the same expression in the right side is

$$\begin{aligned} &= \sum_{T_{q_{s,1}} \cup T_{q-1} = T} \left[ \frac{(r-1)!}{e!(r-e-1)!} \cdot \frac{e!}{(q-1)!} \cdot \frac{(r-e-1)!}{(q_{s,1})!} \right. \\ &\quad \left. \cdot \frac{1}{(i_1!)^{m_1} \dots (i_{p'})^{m_{p'}}} \cdot P(T_{q_{s,1}}) \cdot P(T_{q-1}) \right] \\ &= \frac{(r-1)!}{(q-1)!(q_{s,1})!} \cdot \frac{1}{(i_1!)^{m_1} \dots (i_{p'})^{m_{p'}}} \sum_{T_{q_{s,1}} \cup T_{q-1} = T} P(T_{q_{s,1}}) \cdot P(T_{q-1}). \end{aligned}$$

Which is obtained by noticing that for each possible value of  $e$  we have  $q_{s,1} = q_{s',1}^{r-e}$  for some  $s'$ .

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